THE HODGE STRUCTURE OF SEMIAMPLE HYPERSURFACES AND A GENERALIZATION OF THE MONOMIAL-DIVISOR MIRROR MAP

ANVAR R. MAVLYUTOV

ABSTRACT. We explicitly compute the cohomology ring of semiample nondegenerate hypersurfaces in complete simplicial toric varieties. The monomial-divisor mirror map is generalized to a map between the whole Picard group and the space of infinitesimal deformations for a mirror pair of Calabi-Yau hypersurfaces. This map is compatible with certain vanishing limiting products of the subrings of the chiral rings, on which the ring structure is related to a product of the roots of A-type Lie algebra.

CONTENTS

In this paper we solve the long-standing problem of describing the cohomology of semiample hypersurfaces in complete simplicial toric varieties. The problem became important when physicists found a relation between quantum field theory and the geometry of Calabi-Yau manifolds. Toric varieties used as an ambient space provide a large number of examples of Calabi-Yau manifolds realized as complete intersections. Cohomology of Calabi-Yau manifolds is interpreted as a finite dimensional subspace of the Hilbert space of states in physics. Physicists used the (mathematically non-rigorous) orbifold Landau-Ginzburg theory to compute the cohomology of Calabi-Yau manifolds in some examples coming from toric varieties (e.g., [V]). However, no general solution in physics was found. About ten years ago physicists discovered Mirror Symmetry, which started with a certain symmetry of the Hodge numbers of two topologically distinct Calabi-Yau manifolds. Soon after that, V. Batyrev in [B] generalized the construction of physicists using a duality of reflexive polytopes and Calabi-Yau hypersurfaces in toric varieties. Being the

resolutions of ample hypersurfaces, these mirror symmetric hypersurfaces are semi-ample. The problem to compute the cohomology is also relevant to the calculation of the B-model chiral ring since the cohomology is isomorphic to this ring.

In most discussions of the Mirror Theorems (see [CKa]), it is not mentioned that the results proved by Givental and Lian-Liu-Yau apply only to some (called toric and polynomial) parts of the A-model and B-model chiral rings of a mirror pair of Calabi-Yau manifolds. An explicit description of the full cohomology of the manifolds is a necessary part to extend their techniques to a complete solution.

The history of calculating the cohomology of hypersurfaces starts with a paper of P. Griffiths (see [G]). There he found the residue (primitive) part of the middle cohomology of smooth hypersurfaces in a projective space. Such hypersurfaces are automatically ample, implying certain vanishing theorems which allow to do the calculations. In a later paper [CaG], the product structure was computed on the residue part. Using essentially the same method, V. Batyrev and D. Cox in [BC] described the residue part of the middle cohomology of ample quasismooth hypersurfaces in complete simplicial toric varieties, for which the analogous vanishing theorems hold as well. For semiample hypersurfaces the vanishing theorems do not hold – that is where the complexity of the problem arises. However, the paper [BC] also contained an alternative method of finding the description for an open subset of ample quasismooth hypersurfaces which consists of nondegenerate (transversal to the torus orbits) hypersurfaces. In this case, the Gysin spectral sequence together with the ampleness have been used. We noticed that this spectral sequence can be applied to the semiample nondegenerate hypersurfaces as well, but because of the missing ampleness property the complexity still appears. While this paper provides a solution only for the semiample nondegenerate hypersurfaces, we said in [M3] that our strategy is to use this solution in order to find an answer to the question for all semiample quasismooth hypersurfaces. The answer should be very similar since the semiample quasismooth hypersurfaces are flat deformations of the nondegenerate ones.

Here is the plan of our paper. In Section 1, we start with introducing the notation, definitions, and then review some basic results from [M2].

Section 2 is devoted to deriving our main result: an explicit description of the cohomology ring of semiample nondegenerate hypersurfaces in complete simplicial toric varieties. In [M2] we showed that this cohomology splits into the toric and residue parts. The toric part is the image of cohomology of the ambient space, while its complement comes from the residues of rational differential forms with poles along the hypersurface. The toric part of the cohomology has already been calculated, and we follow the algorithmic approach suggested in [M2] to derive the residue part. In parallel, we also compute the product structure on the cohomology.

In Section 3, we apply the results from Section 2 to find the Picard group of semiample nondegenerate hypersurfaces. We give two alternative descriptions of the Picard group, one of which is algebraic and the other is geometric. A previous approach of [R] to compute the Picard group has gaps in the proof, complicated restrictions and not quite explicit (the toric variety $\mathbb{P}_{(\Sigma_0, L)}$ in the proof of [R, Theorem 2] is not necessarily simplicial, hence [R, Proposition 4] could not be used in this case).

In Section 4, we use one of the main results from [M2] to describe a subring of $H^*(X, \wedge^* \mathcal{T}_X)$ of semiample minimal Calabi-Yau hypersurfaces in toric varieties,

which contains $H^1(X, \mathcal{T}_X)$. Here, $\wedge^* \mathcal{T}_X$ is the *Zariski p-th exterior power of the tangent sheaf*, introduced in [M2]. We discover that the remarkable product structure on the subring is closely related to a product of the roots of an A-type Lie algebra. Looking at the subring of the cohomology of semiample minimal Calabi-Yau hypersurfaces, generated by the Picard group, we find that a very similar product structure takes place.

Using the above, in Section 5, we propose a generalization of the monomial-divisor mirror map of Aspinwall, Greene and Morrison, which is supposed to be the derivative of the mirror map at the large radius limit points. Our map is supported by a compatibility of certain vanishing limiting products of the chiral rings of Calabi-Yau hypersurfaces in the Batyrev mirror construction. We also predict that the same vanishing occurs in the quantum cohomology of the Calabi-Yau hypersurfaces.

Acknowledgments. While all the essential techniques for solving this problem were developed in my Ph.D. dissertation (see [M3]), I found the last clue this summer being a Liftoff researcher of the Clay Mathematics Institute. I would like to thank David Cox for suggesting me this complicated and interesting problem for my dissertation research, and, also, for many helpful discussions and useful comments.

1. A BRIEF REVIEW OF SEMIAMPLE DIVISORS IN TORIC VARIETIES.

In this section we review some basic notation and facts from [M2] about semiample divisors in toric varieties.

Throughout this paper we will use the following dictionary:

M is a lattice of rank d.

$N = \mathrm{Hom}(M, \mathbb{Z})$ is the dual lattice.

$M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ are the $\mathbb{R}$-scalar extensions of M and N, respectively.

Σ is a finite rational (usually simplicial) fan in $N_{\mathbb{R}}$.

$\mathbf{P}_{\Sigma}$ is a d-dimensional toric variety associated with Σ.

$\mathbf{T}_{\sigma}$ is a torus corresponding to the cone $\sigma \in \Sigma$.

$V(\sigma)$ is a toric variety equal to the closure of $\mathbf{T}_{\sigma}$ in $\mathbf{P}_{\Sigma}$.

$\Sigma(k)$ is the set of all k-dimensional cones in Σ.

$\Sigma(1) = \{\rho_1, \ldots, \rho_n\}$ is the set of 1-dimensional cones in Σ with the minimal integral generators $e_1, \ldots, e_n$, respectively.

D_i is a torus invariant divisor in $\mathbf{P}_{\Sigma}$, corresponding to ρ_i .

$S = S(\mathbf{P}_{\Sigma}) = \mathbb{C}[x_1, \ldots, x_n]$ is the homogeneous coordinate ring of $\mathbf{P}_{\Sigma}$.

$\Delta_D = \{m \in M_{\mathbb{R}} : \langle m, e_i \rangle \geq -a_i \text{ for all } i\} \subset M_{\mathbb{R}}$ is a convex polytope of a Weil divisor $D = \sum_{i=1}^n a_i D_i$.

A Cartier divisor D on $\mathbf{P}_{\Sigma}$ is called *semiample* if $\mathcal{O}_{\mathbf{P}_{\Sigma}}(D)$ is generated by global sections. The Iitaka dimension of a Cartier divisor D on $\mathbf{P}_{\Sigma}$ is defined by

$$\kappa(D) := \kappa(\mathcal{O}_{\mathbf{P}_{\Sigma}}(D)) = \dim \phi_D(\mathbf{P}_{\Sigma}),$$

where $\phi_D : \mathbf{P}_{\Sigma} \to \mathbb{P}(H^0(\mathbf{P}_{\Sigma}, \mathcal{O}_{\mathbf{P}_{\Sigma}}(D)))$ is the rational map defined by the sections of the line bundle $\mathcal{O}_{\mathbf{P}_{\Sigma}}(D)$. For a torus invariant D, $\kappa(D)$ coincides with the dimension of the associated polytope Δ_D.

Definition 1.1. [M2] A semiample divisor D on a complete toric variety $\mathbf{P}_{\Sigma}$ is called *i-semiample* if the Iitaka dimension $\kappa(D) = i$.

The following result which describes the unique properties of the map associated with a semiample divisor is a simplification of Theorem 1.4 in [M2].

Theorem 1.2. *Let $[D] \in A_{d-1}(\mathbf{P}_\Sigma)$ be an i-semiample divisor class on a complete toric variety $\mathbf{P}_\Sigma$ of dimension d. Then, there exists a unique complete toric variety $\mathbf{P}_{\Sigma_D}$ with a surjective morphism $\pi : \mathbf{P}_\Sigma \to \mathbf{P}_{\Sigma_D}$, arising from a surjective homomorphism of lattices $\tilde{\pi} : N \to N_D$ which maps the fan Σ into Σ_D, such that $\pi^*[Y] = [D]$ for some ample divisor Y on $\mathbf{P}_{\Sigma_D}$. Moreover, $\dim \mathbf{P}_{\Sigma_D} = i$, and the fan Σ_D is the normal fan of Δ_D for a torus invariant D.*

Remark 1.3. The fan Σ_D lies in the space $(N_D)_\mathbb{R} := N_\mathbb{R}/N'_\mathbb{R}$, where $N' = \{v \in N : \psi_D(-v) = -\psi_D(v)\}$ is a sublattice of N and ψ_D is the support function associated with D. Also, Δ_D lies in $(M_D)_\mathbb{R}$, where $M_D := N'^\perp \cap M$ (for details see [M2, Section 1]).

Remark 1.4. The map π in the above theorem is well known and the variety $\mathbf{P}_{\Sigma_D}$ can be represented as $\mathbf{Proj}\big(\oplus_{k \geq 0} H^0(\mathbf{P}_\Sigma, \mathcal{O}_{\mathbf{P}_\Sigma}(kD))\big)$. The condition on the map says that it is surjective with connected fibers. It would be interesting to see an analogue of the above theorem in a broader context.

We will need to use the following intersection properties of semiample divisors.

Lemma 1.5. *If D is an i-semiample divisor on a complete toric variety $\mathbf{P}_\Sigma$, then the intersection number $(D^k \cdot V(\tau)) > 0$ for any $\tau \in \Sigma(d-k)$, such that $\tilde{\pi}(\tau)$ is contained in a cone of $\Sigma_D(i-k)$, and $(D^k \cdot V(\tau)) = 0$ for all other $\tau \in \Sigma(i-k)$.*

Proof. The arguments are the same as for Lemma 1.4 in [M1]. $\qquad\qquad\square$

Let D be a semiample (torus invariant) divisor in degree $\beta \in A_{d-1}(\mathbf{P}_\Sigma)$ for a complete toric variety $\mathbf{P}_\Sigma$. By Theorem 1.2, we have the associated toric morphism $\pi : \mathbf{P}_\Sigma \to \mathbf{P}_{\Sigma_D}$. Let also $\sigma \in \Sigma_X$ be the smallest cone , containing the image of the cone $\tau \in \Sigma$, and $\sigma' \in \Sigma(d - i + \dim \sigma)$ be such that $\tau \subset \sigma'$ and $\tilde{\pi}(\sigma') \subset \sigma$. From [M2, Section 1] it follows that there is a natural commutative diagram:

$$
\begin{array}{ccccc}
S(\Sigma)_{p\beta} & \xrightarrow{\varphi^*_{\sigma'}} & S(V(\sigma'))_{p\beta\sigma'} & \xrightarrow{\pi_{\sigma'*}} & S(V(\sigma))_{p\beta\sigma} \\
\downarrow & & \downarrow & & \downarrow \\
\end{array}
$$

$$H^0(\mathbf{P}_\Sigma, \mathcal{O}_{\mathbf{P}_\Sigma}(pD)) \xrightarrow{\varphi^*_{\sigma'}} H^0(V(\sigma'), \mathcal{O}_{V(\sigma')}(pD_{\sigma'})) \xrightarrow{\pi_{\sigma'*}} H^0(V(\sigma), \mathcal{O}_{V(\sigma)}(p\pi_*D_{\sigma'})),$$

where $\beta^{\sigma'} = [D_{\sigma'}]$, $D_{\sigma'} := D|_{V(\sigma')} = D \cdot V(\sigma')$, in the Chow group of $V(\sigma')$, $\beta^\sigma = \pi_*[D_{\sigma'}]$, $\varphi^*_{\sigma'}$ is the restriction of the global sections, the map $\pi_{\sigma'*}$ is a pushforward of the sections, and the vertical arrows are isomorphisms.

In this paper, we are interested in the *nondegenerate* hypersurfaces $X \subset \mathbf{P}_\Sigma$, which have only transversal intersections with the tori $\mathbf{T}_\sigma$. This is equivalent to saying that $x_1 \partial f / \partial x_1, \ldots, x_n \partial f / \partial x_n$ do not vanish simultaneously on $\mathbf{P}_\Sigma$, where $f \in S_\beta$ determines the hypersurface X. A generic hypersurface in a given semiample degree is nondegenerate, by [D, Proposition 6.8]. We have the following property for such hypersurfaces.

Proposition 1.6. [M2] *Let X be a semiample nondegenerate hypersurface in a complete toric variety $\mathbf{P}_\Sigma$, and let $\pi : \mathbf{P}_\Sigma \to \mathbf{P}_{\Sigma_X}$ be the associated morphism for $[X] \in A_{d-1}(\mathbf{P}_\Sigma)$, then $Y = \pi(X)$ is a nondegenerate ample hypersurface, and $X = \pi^{-1}(Y)$.*

For the affine hypersurfaces in tori, there is a Lefschetz-type theorem.

Theorem 1.7. *Let X be an i-semiample nondegenerate hypersurface in a complete toric variety $\mathbf{P}_\Sigma$. Then the natural restrictions*

$$H^l(\mathbf{T}) \to H^l(X \cap \mathbf{T})$$

are isomorphisms for $l < i - 1$, where $\mathbf{T} \subset \mathbf{P}_\Sigma$ is the maximal dimensional torus.

Proof. This follows from the corresponding result (see [DK]), when the hypersurface is ample, equation (2) in [M1] and the Künneth isomorphism. $\square$

In [M2, Section 5], we proved that the cohomology of a semiample nondegenerate hypersurface X in a complete simplicial toric variety $\mathbf{P}_\Sigma$ has a direct sum decomposition:

$$H^*(X) = H^*_{\text{toric}}(X) \oplus H^*_{\text{res}}(X),$$

where $H^*_{\text{toric}}(X) := \text{im}(H^*(\mathbf{P}_\Sigma) \xrightarrow{i^*} H^*(X))$ and $H^*_{\text{res}}(X) := \text{im}(H^{*+1}(\mathbf{P}_\Sigma \backslash X) \xrightarrow{\text{Res}} H^*(X))$ are called the *toric* and *residue* parts of the cohomology. These parts are orthogonal to each other:

$$\int_X H^*_{\text{toric}}(X) \cup H^*_{\text{res}}(X) = 0.$$

In [M2, Theorem 5.1], we also calculated

$$H^*_{\text{toric}}(X) \cong H^*(\mathbf{P}_\Sigma)/Ann([X])$$

where $Ann([X])$ is the annihilator of the class $[X] \in H^2(\mathbf{P}_\Sigma)$. The cohomology of $\mathbf{P}_\Sigma$ is isomorphic to

$$\mathbb{C}[D_1, \ldots, D_n]/(P(\Sigma) + SR(\Sigma)),$$

where the generators correspond to the torus invariant divisors of $\mathbf{P}_\Sigma$, and where

$$P(\Sigma) = \left\langle \sum_{i=1}^n \langle m, e_i \rangle D_i : m \in M \right\rangle,$$

$$SR(\Sigma) = \left\langle D_{i_1} \cdots D_{i_k} : \{e_{i_1}, \ldots, e_{i_k}\} \not\subset \sigma \text{ for all } \sigma \in \Sigma \right\rangle.$$

Hence, $H^*_{\text{toric}}(X)$ is isomorphic to the bigraded ring

$$A_1(X)_{*,*} := \mathbb{C}[D_1, \ldots, D_n]/I,$$

where $I = (P(\Sigma) + SR(\Sigma)) : [X]$ is the ideal quotient, and D_k have the degree $(1,1)$.

We finish this section with some additional frequently used notation:

$S(V(\tau))$, for $\tau \in \Sigma$, is the homogeneous coordinate ring of $V(\tau)$.

$\varphi_\tau : X \cap V(\tau) \subset X$ is the inclusion for $\tau \in \Sigma$.

$\varphi_{\tau,\gamma} : X \cap V(\gamma) \subset X \cap V(\tau)$ is the inclusion for $\tau, \gamma \in \Sigma$ such that $\tau \subset \gamma$.

$f_\tau = \varphi_\tau^*(f) \in S(V(\tau))$ denotes the polynomial defining $X \cap V(\tau)$ for $\tau \in \Sigma$.

$\text{int}\tau = \tau \setminus \left(\bigcup_{\gamma \prec \tau} \gamma\right)$ is the relative interior of a cone τ, where the union is by all proper faces of τ.

$\pi_\tau : V(\tau) \to V(\sigma)$ is the restriction of π for $\tau \in \Sigma$ and $\sigma \in \Sigma_X$ such that $\tilde{\pi}(\text{int}\tau) \subset \text{int}\sigma$.

$f_\sigma = \pi_{\tau*}(f_\tau) \in S(V(\sigma))$ is the polynomial defining the hypersurface $\pi(X) \cap V(\sigma)$ in $V(\sigma)$.

$d(\tau) := d - \dim \tau$ for $\tau \in \Sigma$, and $i(\sigma) := i - \dim \sigma$ for $\sigma \in \Sigma_X$.

$\Gamma_\sigma \subset \Delta_D$ is the face corresponding to $\sigma \in \Sigma_D$.

$\beta_1^\sigma = \deg(\prod_{\tilde{\pi}(\rho_k) \subset \sigma} x_k) \in A_{d-1}(\mathbf{P}_\Sigma)$ for $\sigma \in \Sigma_X$.

2. The Hodge structure of semiample nondegenerate hypersurfaces.

The goal of this section is to compute the residue part of the cohomology of semiample nondegenerate hypersurfaces. This will completely generalize our previous results in [M1] and [M2], concerning the middle cohomology of big and nef hypersurfaces. The essential idea comes from the algorithmic approach in [M2, Section 5] to computing of the residue part of cohomology. In fact, our proof will use an induction, where [M1, Theorem 4.4] may be considered as the base for this induction. This allows us to get an explicit description of the generators of cohomology. Quite remarkably, the Poincaré duality does the rest of the job: it gives all the relations among the generators.

Let X be an i-semiample nondegenerate hypersurface in a complete simplicial toric variety $\mathbf{P}_\Sigma$, and let $\pi : \mathbf{P}_\Sigma \to \mathbf{P}_{\Sigma_X}$ be the associated contraction. From the diagram (14) in [M2], we have an exact sequence:

$$\bigoplus_{k=1}^{n} H_{\mathrm{res}}^{s-2}(X \cap D_k) \xrightarrow{\oplus \varphi_{\rho_k!}} H_{\mathrm{res}}^s(X) \to \mathrm{Gr}_s^W PH^s(X \cap \mathbf{T}) \to 0.$$

A similar sequence can be written for $X \cap D_k \subset D_k$ instead of $X \subset \mathbf{P}_\Sigma$ as well as for all hypersurfaces $X \cap V(\tau)$ in the toric subvarieties $V(\tau)$:

$$\bigoplus_{\tau \subset \tau' \in \Sigma(\dim \tau + 1)} H_{\mathrm{res}}^{s-2}(X \cap V(\tau')) \to H_{\mathrm{res}}^s(X \cap V(\tau)) \to \mathrm{Gr}_s^W PH^s(X \cap \mathbf{T}_\tau) \to 0. \tag{1}$$

Our first step is to show how to split the residue part $H_{\mathrm{res}}^s(X \cap V(\tau))$. Then it will be clear how to obtain a description of $H_{\mathrm{res}}^s(X)$ from the above exact sequences. We closely follow the arguments in [M2, Section 6]. By the construction of the fan Σ_X there is the smallest cone $\sigma \in \Sigma_X$, containing the image of the cone $\tau \in \Sigma$. The map π restricted to the toric subvariety $V(\tau) \subset \mathbf{P}_\Sigma$ gives the contraction $\pi_\tau : V(\tau) \to V(\sigma)$. We will need to verify some information with a help of a simplicial toric subvariety $V(\sigma') \subset V(\tau)$ associated with a cone $\sigma' \in \Sigma(d-i+\dim \sigma)$ such that $\tau \subset \sigma'$ and the image of σ' is in σ. In this case, the toric variety $V(\sigma')$ is mapped birationally onto $V(\sigma)$, and we get a commutative diagram

$$
\begin{array}{ccc}
H^0(V(\sigma), \Omega_{V(\sigma)}^{i(\sigma)}(*Y_\sigma)) & \xrightarrow{\;\pi_\tau^*\;} & H^0(V(\tau), \Omega_{V(\tau)}^{i(\sigma)}(*X_\tau)) \\
& \searrow{\scriptstyle \pi_{\sigma'}^*} & \Big\downarrow{\scriptstyle \varphi_{\tau,\sigma'}^*} \\
& & H^0(V(\sigma'), \Omega_{V(\sigma')}^{i(\sigma)}(*X_{\sigma'})),
\end{array}
\tag{2}
$$

where we use short notations: $i(\sigma) = i - \dim \sigma$, $Y_\sigma = Y \cap V(\sigma)$, $X_\tau = X \cap V(\tau)$, $X_{\sigma'} = X \cap V(\sigma')$. There is a natural map

$$S(V(\sigma))_{(p+1)\beta^\sigma - \beta_0^\sigma} \to H^0(V(\sigma), \Omega_{V(\sigma)}^{i(\sigma)}((p+1)Y_\sigma))$$

sending a polynomial A to the form $A\Omega_{V(\sigma)}/f_\sigma^{p+1}$, where $\Omega_{V(\sigma)}$ is the $i(\sigma)$-form for the complete toric variety $V(\sigma)$, corresponding to a basis $m_1^\sigma, \ldots, m_{i(\sigma)}^\sigma$ of the lattice $M_X \cap \sigma^\perp$ (see [BC, Definition 9.3]). The complete toric variety $V(\sigma)$ has homogeneous coordinates y_γ corresponding to the integral generators e_γ of the 1-dimensional cones of the fan of $V(\sigma)$. We want to compute the pull-back of the form $A\Omega_{V(\sigma)}/(f_\sigma)^{p+1}$ with respect to the maps in (2). Since the form vanishes

outside the torus $\mathbf{T}_\sigma$, the calculation can be done using the affine coordinates

$$t_j^\sigma = \prod_\gamma y_\gamma^{\langle m_j^\sigma, e_\gamma \rangle}, \quad j = 1, \ldots, i(\sigma),$$

on the torus. Under the map $\pi_{\sigma'}$, the coordinates t_j^σ are identified with the monomials $\prod_{\gamma'} x_{\gamma'}^{\langle m_j^\sigma, e_{\gamma'} \rangle}$, where $x_{\gamma'}$, for $\sigma' \subset \gamma' \in \Sigma(\dim(\sigma') + 1)$, are the homogeneous coordinates on $V(\sigma')$, corresponding to the integral generators $e_{\gamma'}$. Note

$$\Omega_{V(\sigma)} = \prod_\gamma y_\gamma \left(\sum_\gamma \langle m_1^\sigma, e_\gamma \rangle \frac{dy_\gamma}{y_\gamma} \right) \wedge \cdots \wedge \left(\sum_\gamma \langle m_{i(\sigma)}^\sigma, e_\gamma \rangle \frac{dy_\gamma}{y_\gamma} \right)$$

$$= \prod_\gamma y_\gamma \frac{dt_1^\sigma}{t_1^\sigma} \wedge \cdots \wedge \frac{dt_{i(\sigma)}^\sigma}{t_{i(\sigma)}^\sigma}.$$

Therefore, we get

$$\pi_{\sigma'}^* \left(\frac{A\Omega_{V(\sigma)}}{f_\sigma^{p+1}} \right) = \pi_{\sigma'}^* \left(\frac{A \prod_\gamma y_\gamma}{f_\sigma^{p+1}} \right) \frac{dt_1^\sigma}{t_1^\sigma} \wedge \cdots \wedge \frac{dt_{i(\sigma)}^\sigma}{t_{i(\sigma)}^\sigma}$$

$$= \frac{\pi_{\sigma'}^*(A \prod_\gamma y_\gamma)}{f_{\sigma'}^{p+1}} \left(\sum_{\gamma'} \langle m_1^\sigma, e_{\gamma'} \rangle \frac{dx_{\gamma'}}{x_{\gamma'}} \right) \wedge \cdots \wedge \left(\sum_{\gamma'} \langle m_{i(\sigma)}^\sigma, e_{\gamma'} \rangle \frac{dx_{\gamma'}}{x_{\gamma'}} \right)$$

$$= \frac{\pi_{\sigma'}^*(A \prod_\gamma y_\gamma)}{(\prod_{\gamma'} x_{\gamma'}) f_{\sigma'}^{p+1}} \Omega_{V(\sigma')},$$

where $\Omega_{V(\sigma')}$ is the $i(\sigma)$-form for the complete simplicial toric variety $V(\sigma')$, corresponding to the basis $m_1^\sigma, \ldots, m_{i(\sigma)}^\sigma$ of the lattice $M \cap \sigma'^\perp = M_X \cap \sigma^\perp$ (see [BC, Definition 9.3]).

Similarly to the above, we can also compute the pull-back π_τ^* of the form $A\Omega_{V(\sigma)}/f_\sigma^{p+1}$. But now the coordinates t_j^σ can be pulled by the map π_τ to the torus $\mathbf{T}_\tau$ and expressed in terms of the homogeneous coordinates of $\mathbf{P}_\Sigma$:

$$t_j^\sigma = \prod_{k=1}^n x_k^{\langle m_j^\sigma, e_k \rangle}.$$

Completing $m_1^\sigma, \ldots, m_{i(\sigma)}^\sigma$ to a basis of the lattice $M \cap \tau^\perp$, we get a full set of coordinates $t_1^\sigma, \ldots, t_{i(\sigma)}^\sigma, t_{i(\sigma)+1}^\sigma, \ldots, t_{d-\dim \tau}^\sigma$ on the torus $\mathbf{T}_\tau$. Changing the basis, if necessary, we may also assume that these coordinates are well defined on the other tori which map onto $\mathbf{T}_\sigma$. On all such tori the map π_τ is given by projecting to the first $i(\sigma)$ affine coordinates. Consequently, we get

$$\pi_\tau^* \left(\frac{A\Omega_{V(\sigma)}}{f_\sigma^{p+1}} \right) = \pi_\tau^* \left(\frac{A \prod_\gamma y_\gamma}{f_\sigma^{p+1}} \right) \frac{dt_1^\sigma}{t_1^\sigma} \wedge \cdots \wedge \frac{dt_{i(\sigma)}^\sigma}{t_{i(\sigma)}^\sigma}$$

$$= \frac{\pi_\tau^*(A \prod_\gamma y_\gamma)}{f^{p+1}} \left(\sum_{k=1}^n \langle m_1^\sigma, e_k \rangle \frac{dx_k}{x_k} \right) \wedge \cdots \wedge \left(\sum_{k=1}^n \langle m_{i(\sigma)}^\sigma, e_k \rangle \frac{dx_k}{x_k} \right)$$

$$= \frac{\pi_\tau^*(A \prod_\gamma y_\gamma)}{(\prod_{\tilde{\pi}(\rho_k) \not\subset \sigma} x_k) f^{p+1}} \Omega_\sigma,$$

where we introduce an $i(\sigma)$-form on the toric variety $V(\tau)$:

$$\Omega_\sigma := \prod_{\tilde{\pi}(\rho_k) \not\subset \sigma} x_k \left(\sum_{k=1}^n \langle m_1^\sigma, e_k \rangle \frac{\mathrm{d}x_k}{x_k} \right) \wedge \cdots \wedge \left(\sum_{k=1}^n \langle m_{i(\sigma)}^\sigma, e_k \rangle \frac{\mathrm{d}x_k}{x_k} \right).$$

Here, we slightly abused the notation using the identification

$$\left(S/\langle x_k : \tilde{\pi}(\rho_k) \subset \sigma \rangle \right)_{(p+1)\beta} \cong S(V(\tau))_{(p+1)\beta},$$

which follows the same way as the equation (17) in [M2].

From the above calculations we can see a commutative diagram:

$$
\begin{array}{ccc}
S_{(p+1)\beta-\beta_0+\beta_1^\sigma} & \longrightarrow & H^0(V(\tau), \Omega_{V(\tau)}^{i(\sigma)}((p+1)X_\tau)) \\
\downarrow{\varphi_{\sigma'}^*} & & \downarrow{\varphi_{\tau,\sigma'}^*} \\
S(V(\sigma'))_{(p+1)\beta^{\sigma'}-\beta_0^{\sigma'}} & \longrightarrow & H^0(V(\sigma'), \Omega_{V(\sigma')}^{i(\sigma)}((p+1)X_{\sigma'})),
\end{array}
$$

where $\beta_1^\sigma = \deg(\prod_{\tilde{\pi}(\rho_k) \subset \sigma} x_k)$ and the horizontal arrows are represented by the forms

$$\frac{B\Omega_\sigma}{f^{p+1}} \quad \text{and} \quad \frac{B'\Omega_{V(\sigma')}}{f_{\sigma'}^{p+1}}$$

for the corresponding polynomials B and B', while the arrow on the left is defined by

$$\varphi_{\sigma'}^*(B) := \frac{\varphi_{\sigma'}^*(B \prod_{\tilde{\pi}(\rho_k) \not\subset \sigma} x_k)}{\prod_{\gamma'} x_{\gamma'}}.$$

Applying the residue map, we get another diagram:

$$
\begin{array}{ccc}
S_{(p+1)\beta-\beta_0+\beta_1^\sigma} & \xrightarrow{\mathrm{Res}_\tau} & H^p(X \cap V(\tau), \Omega_{X\cap V(\tau)}^{i(\sigma)-1-p}) \\
\downarrow{\varphi_{\sigma'}^*} & & \downarrow{\varphi_{\tau,\sigma'}^*} \\
S(V(\sigma'))_{(p+1)\beta^{\sigma'}-\beta_0^{\sigma'}} & \xrightarrow{\mathrm{Res}_{\sigma'}} & H^p(X \cap V(\sigma'), \Omega_{X\cap V(\sigma')}^{i(\sigma)-1-p}).
\end{array}
\tag{3}
$$

By [M1, Theorem 4.4], we know that the bottom arrow induces a well defined map on $R_1(f_{\sigma'})_{(p+1)\beta^{\sigma'}-\beta_0^{\sigma'}}$. To show a similar result for the top arrow, as in [M2, Definition 6.5], we introduce the following rings.

Definition 2.1. Given $f \in S_\beta$ of semiample degree $\beta = [D] \in A_{d-1}(\mathbf{P}_\Sigma)$ and $\sigma \in \Sigma_D$, let $J_0^\sigma(f)$ be the ideal in S generated by the ideal $J_0(f)$ and all x_k such that $\tilde{\pi}(\rho_k) \subset \sigma$, and let $J_1^\sigma(f)$ be the ideal quotient $J_0^\sigma(f) : (\prod_{\tilde{\pi}(\rho_k) \not\subset \sigma} x_k)$. Then we get the quotient rings $R_0^\sigma(f) = S/J_0^\sigma(f)$ and $R_1^\sigma(f) = S/J_1^\sigma(f)$ graded by the Chow group $A_{d-1}(\mathbf{P}_\Sigma)$.

Proposition 2.2. *Let $\beta = [D] \in A_{d-1}(\mathbf{P}_\Sigma)$ be i-semiample, and let $\sigma \in \Sigma_D$ and $\sigma' \in \Sigma(d-i+\dim\sigma)$ be such that $\tilde{\pi}(\sigma') \subset \sigma$. If $\beta_0^{\sigma'} = \deg(\prod_{\gamma'} x_{\gamma'}) \in A_{i(\sigma)-1}(V(\sigma'))$ and $\beta_0^\sigma = \deg(\prod_\gamma y_\gamma) \in A_{i(\sigma)-1}(V(\sigma))$ denote the anticanonical degrees, then, there are natural isomorphisms induced by $\varphi_{\sigma'}^*$ and $\pi_{\sigma'*}$:*
  (i) $R_0^\sigma(f)_{*\beta} \cong R_0(f_{\sigma'})_{*\beta^{\sigma'}} \cong R_0(f_\sigma)_{*\beta^\sigma}$,
  (ii) $R_1^\sigma(f)_{*\beta-\beta_0+\beta_1^\sigma} \cong R_1(f_{\sigma'})_{*\beta^{\sigma'}-\beta_0^{\sigma'}} \cong R_1(f_\sigma)_{*\beta^\sigma-\beta_0^\sigma}$.
  (iii) $R_0^\sigma(f)_{*\beta-\beta_0+\beta_1^\sigma} \cong R_0(f_{\sigma'})_{*\beta^{\sigma'}-\beta_0^{\sigma'}} \cong R_0(f_\sigma)_{*\beta^\sigma-\beta_0^\sigma}$.

Proof. This is just a generalization of Proposition 6.6 in [M2], and the proof follows the same way. $\qquad\square$

In order to prove that the top arrow in (3) is well defined on $R_1^\sigma(f)_{(p+1)\beta-\beta_0+\beta_1^\sigma}$ we will need to use an explicit representation of the $(i(\sigma)-1-p,p)$ Hodge component of the residue $\mathrm{Res}(B\Omega_\sigma/f^{p+1}) \in H^{i(\sigma)-1}(X \cap V(\tau))$.

Proposition 2.3. *Let $X \subset \mathbf{P}_\Sigma$ be an i-semiample nondegenerate hypersurface defined by $f \in S_\beta$. Given $\tau \in \Sigma$ and the smallest cone $\sigma \in \Sigma_X$, containing the image of τ, then, under the natural map*

$$\check{H}^p(\mathcal{U}|_{X\cap V(\tau)}, \Omega_{X\cap V(\tau)}^{i(\sigma)-1-p}) \to H^p(X \cap V(\tau), \Omega_{X\cap V(\tau)}^{i(\sigma)-1-p}) \cong H^{i(\sigma)-1-p,p}(X \cap V(\tau)),$$

the Hodge component of the residue $\mathrm{Res}_\tau(A\Omega_\sigma/f^{p+1}) = \mathrm{Res}(A\Omega_\sigma/f^{p+1})^{i(\sigma)-1-p,p}$, for $A \in S_{(p+1)\beta-\beta_0+\beta_1^\sigma}$, is represented by the Čech cocycle

$$\frac{(-1)^{i(\sigma)-1+(p(p+1)/2)}}{p!}\left\{\frac{AK_{i_p}\cdots K_{i_0}\Omega_\sigma}{f_{i_0}\cdots f_{i_p}}\right\}_{i_0\dots i_p} \in C^p(\mathcal{U}|_{X\cap V(\tau)}, \Omega_{X\cap V(\tau)}^{i(\sigma)-1-p}),$$

where $\mathcal{U} = \{U_i\}_{i=1}^n$ with $U_i = \{x \in \mathbf{P}_\Sigma : x_i f_i(x) \neq 0\}$, $f_i := \partial f/\partial x_i$, is the open cover of $\mathbf{P}_\Sigma$, and where K_i is the contraction operator $\frac{\partial}{\partial x_i}\lrcorner$.

Proof. This proposition generalizes our previous results (see [M1, Theorem 3.3] and [M2, Proposition 7.2]) as well as the corresponding results in [CaG]. However, here we suggest another way to do calculations. There is no need to plunge into hypercohomology as it was done in [CaG]. Since we are interested only in the particular Hodge component of the residue, the two exact sequences

$$0 \to \Omega_{V(\tau)}^{i(\sigma)-p}(\log X_\tau) \to \Omega_{V(\tau)}^{i(\sigma)-p}(X_\tau) \xrightarrow{\mathrm{d}} \Omega_{V(\tau)}^{i(\sigma)-p+1}(2X_\tau)/\Omega_{V(\tau)}^{i(\sigma)-p+1}(X_\tau) \xrightarrow{\mathrm{d}}$$

$$\cdots \xrightarrow{\mathrm{d}} \Omega_{V(\tau)}^{d(\tau)}((d(\tau)-i(\sigma)+p+1)X_\tau)/\Omega_{V(\tau)}^{d(\tau)}((d(\tau)-i(\sigma)+p)X_\tau) \to 0$$

(here, $d(\tau) := \dim(V(\tau)) = d - \dim\tau$; see [BC, Proposition 10.1]) and

$$0 \to \Omega_{V(\tau)}^{i(\sigma)-p} \to \Omega_{V(\tau)}^{i(\sigma)-p}(\log X_\tau) \xrightarrow{\mathrm{Res}} \Omega_{X_\tau}^{i(\sigma)-p-1} \to 0$$

suffice to produce a representative of the component. Define auxiliary sheaves L_s as the kernels of the morphisms

$$\Omega_{V(\tau)}^{i(\sigma)-p+s-1}(sX_\tau)/\Omega_{V(\tau)}^{i(\sigma)-p+s-1}((s-1)X_\tau) \xrightarrow{\mathrm{d}} \Omega_{V(\tau)}^{i(\sigma)-p+s}((s+1)X_\tau)/\Omega_{V(\tau)}^{i(\sigma)-p+s}(sX_\tau)$$

for $s = 1, \ldots, p+1$. Then we get the short exact sequences

$$0 \to L_s \to \Omega_{V(\tau)}^{i(\sigma)-p+s-1}(sX_\tau)/\Omega_{V(\tau)}^{i(\sigma)-p+s-1}((s-1)X_\tau) \to L_{s+1} \to 0$$

giving rise to the maps in cohomology:

$$H^0(V(\tau), L_{p+1}) \to H^1(V(\tau), L_p) \to$$

$$\cdots \to H^{p-1}(V(\tau), L_2) \to H^p(V(\tau), \Omega_{V(\tau)}^{i(\sigma)-p}(\log X_\tau)). \quad (4)$$

The first part of our proof is to show that the form $A\Omega_\sigma/f^{p+1}$ is a section in $H^0(V(\tau), L_{p+1})$. For this it suffices to check that the differential of this form is a form on $V(\tau)$ with the order of the pole along X_τ at most $p+1$. Since

$$\mathrm{d}\left(\frac{A\Omega_\sigma}{f^{p+1}}\right) = \frac{\mathrm{d}(A\Omega_\sigma)}{f^{p+1}} - \frac{(p+1)A\mathrm{d}f \wedge \Omega_\sigma}{f^{p+2}},$$

the desired property would follow from the fact:

$$\mathrm{d}f \wedge \Omega_\sigma \equiv 0 \quad \text{modulo multiples of } f, x_k \text{ and } \mathrm{d}x_k \text{ for } \tilde{\pi}(\rho_k) \subset \tau. \tag{5}$$

In order to prove this equation, note that Ω_σ is equal up to a multiple by a constant to

$$\frac{x_{l_1} \cdots x_{l_{\dim(\sigma')}} K_{l_{\dim(\sigma')}} \cdots K_{l_1} \Omega}{\prod_{\tilde{\pi}(\rho_k) \subset \sigma} x_k}$$

(where, Ω is the d-form for $\mathbf{P}_\Sigma$ from [BC, Definition 9.3]) for the integral generators $e_{l_1}, \ldots, e_{l_{\dim(\sigma')}}$ of a cone $\sigma' \in \Sigma(d - i + \dim \sigma)$ such that $\tau \subset \sigma'$ and the image of σ' is in σ. This can be seen from the definition of Ω and Ω_σ, and the fact that the choice of a different basis in the definition of Ω affects only the sign of the form. Hence, $\mathrm{d}f \wedge \Omega_\sigma$ is equal up to a multiple by a constant to

$$\mathrm{d}f \wedge \frac{x_{l_1} \cdots x_{l_{\dim(\sigma')}} K_{l_{\dim(\sigma')}} \cdots K_{l_1} \Omega}{\prod_{\tilde{\pi}(\rho_k) \subset \sigma} x_k}$$

$$= (-1)^{\dim(\sigma')} x_{l_1} \cdots x_{l_{\dim(\sigma')}} \frac{K_{l_{\dim(\sigma')}} \cdots K_{l_1} (\mathrm{d}f \wedge \Omega)}{\prod_{\tilde{\pi}(\rho_k) \subset \sigma} x_k}$$

$$+ \sum_{r=1}^{\dim(\sigma')} (-1)^{r-1} f_{l_r} x_{l_1} \cdots x_{l_{\dim(\sigma')}} \frac{K_{l_{\dim(\sigma')}} \cdots \widehat{K_{l_r}} \cdots K_{l_1} \Omega}{\prod_{\tilde{\pi}(\rho_k) \subset \sigma} x_k}.$$

The first summand is a multiple of f by equation (3) in [M1], and we claim that the second is a multiple of x_k or $\mathrm{d}x_k$ for $e_k \in \tau$ such that its image is in the relative interior of σ. Indeed, each summand in Ω is a multiple of x_k or $\mathrm{d}x_k$, while the argument in the proof of Lemma 4.2 in [M2] shows that $x_{l_r} f_{l_r}$ is divisible by all x_k for $\tilde{\pi}(e_k)$ in the relative interior of σ. Thus, equation (5) is proved.

   In the second part, we need to find a representative in the Čech cohomology of the image of the form $A\Omega_\sigma / f^{p+1}$ by the composition of the maps in (4). The form $A\Omega_\sigma / f^{p+1}$ has a representation in $\check{H}^0(\mathcal{U}|_{V(\tau)}, L_{p+1})$ by the Čech cocycle

$$\left\{ \frac{A\mathrm{d}f \wedge K_{i_0}\Omega_\sigma}{f_{i_0} f^{p+1}} \right\}_{i_0},$$

since $\mathrm{d}f \wedge K_{i_0}\Omega_\sigma = f_{i_0}\Omega_\sigma - K_{i_0}(\mathrm{d}f \wedge \Omega_\sigma)$ and because of equation (5). Note that the maps between cohomologies in (4) are the connecting homomorphisms of long exact sequences, so that the image of the Čech cocycle representing $A\Omega_\sigma / f^{p+1}$ can be computed using the following commutative diagrams:

$$0 \longrightarrow C^{p+1-s}(\mathcal{U}|_{V(\tau)}, L_s) \longrightarrow C^{p+1-s}(\mathcal{U}|_{V(\tau)}, \tilde{L}_s) \xrightarrow{\mathrm{d}} C^{p+1-s}(\mathcal{U}|_{V(\tau)}, L_{s+1})$$

$$\uparrow \qquad\qquad\qquad \uparrow \qquad\qquad\qquad \uparrow$$

$$0 \longrightarrow C^{p-s}(\mathcal{U}|_{V(\tau)}, L_s) \longrightarrow C^{p-s}(\mathcal{U}|_{V(\tau)}, \tilde{L}_s) \xrightarrow{\mathrm{d}} C^{p-s}(\mathcal{U}|_{V(\tau)}, L_{s+1}),$$

where $\tilde{L}_s := \Omega_{V(\tau)}^{i(\sigma)-p+s-1}(sX_\tau) / \Omega_{V(\tau)}^{i(\sigma)-p+s-1}((s-1)X_\tau)$ and the vertical arrows are the Čech coboundary maps. When $s = p$, the cocycle representing $A\Omega_\sigma / f^{p+1}$ has a lift to the cochain

$$-\frac{1}{p} \left\{ \frac{AK_{i_0}\Omega_\sigma}{f_{i_0} f^p} \right\}_{i_0} \in C^0(\mathcal{U}|_{V(\tau)}, \Omega_{V(\tau)}^{i(\sigma)-1}(pX_\tau) / \Omega_{V(\tau)}^{i(\sigma)-1}((p-1)X_\tau)),$$

because

$$d\left(\frac{AK_{i_0}\Omega_\sigma}{f_{i_0}f^p}\right) = -pA\frac{\mathrm{d}f \wedge K_{i_0}\Omega_\sigma}{f_{i_0}f^{p+1}} + \frac{\mathrm{d}(AK_{i_0}\Omega_\sigma)}{f_{i_0}f^p} - \frac{A\mathrm{d}f_{i_0} \wedge K_{i_0}\Omega_\sigma}{f^p f_{i_0}^2}.$$

The Čech coboundary of this cochain is

$$-\frac{1}{p}\left\{\frac{AK_{i_1}\Omega_\sigma}{f_{i_1}f^p} - \frac{AK_{i_0}\Omega_\sigma}{f_{i_0}f^p}\right\}_{i_0 i_1} = -\frac{1}{p}\left\{\frac{A(f_{i_0}K_{i_1}\Omega_\sigma - f_{i_1}K_{i_0}\Omega_\sigma)}{f_{i_0}f_{i_1}f^p}\right\}_{i_0 i_1}$$

$$= -\frac{1}{p}\left\{\frac{A(-\mathrm{d}f \wedge K_{i_1}K_{i_0}\Omega_\sigma + K_{i_1}K_{i_0}(\mathrm{d}f \wedge \Omega_\sigma))}{f_{i_0}f_{i_1}f^p}\right\}_{i_0 i_1}$$

$$= \frac{1}{p}\left\{\frac{A\mathrm{d}f \wedge K_{i_1}K_{i_0}\Omega_\sigma}{f_{i_0}f_{i_1}f^p}\right\}_{i_0 i_1},$$

and, by the above diagram, it is a Čech cocycle in $C^1(\mathcal{U}|_{V(\tau)}, L_p)$. Continuing this way, one gets the cocycle

$$\frac{(-1)^p(-1)^{1+\cdots+p}}{p!}\left\{\frac{A\mathrm{d}f \wedge K_{i_p}\cdots K_{i_0}\Omega_\sigma}{f_{i_0}\cdots f_{i_p}f}\right\}_{i_0\ldots i_p}$$

$$= \frac{(-1)^{i(\sigma)-1+(p(p+1)/2)}}{p!}\left\{\frac{AK_{i_p}\cdots K_{i_0}\Omega_\sigma}{f_{i_0}\cdots f_{i_p}} \wedge \frac{\mathrm{d}f}{f}\right\}_{i_0\ldots i_p}$$

in $C^p(\mathcal{U}|_{V(\tau)}, \Omega_{V(\tau)}^{i(\sigma)-p}(\log X_\tau))$, which is ready for application of the residue map. This produces the desired result. □

To compute the cup product on the residue part of cohomology of semiample hypersurfaces, we will need to use certain toric residue maps similar to [C] and [M1]. Let us first recall some facts about toric residues, and then apply them to our situation.

Definition 2.4. [C] Assume $F_0,\ldots,F_d \in S_\beta$ do not vanish simultaneously on a complete d-dimensional toric variety $\mathbf{P}$. Then the *toric residue map*

$$\mathrm{Res}_F : S_\rho/\langle F_0,\ldots,F_d\rangle_\rho \to \mathbb{C},$$

$\rho = (d+1)\beta - \beta_0$, is given by the formula $\mathrm{Res}_F(H) = \mathrm{Tr}_{\mathbf{P}}([\varphi_F(H)])$, where $\mathrm{Tr}_{\mathbf{P}} : H^d(\mathbf{P},\Omega_{\mathbf{P}}^d) \to \mathbb{C}$ is the trace map, and $[\varphi_F(H)]$ is the class represented by the d-form $H\Omega/(F_0\cdots F_d)$ in the Čech cohomology with respect to the open cover $\{x \in \mathbf{P} : F_j(x) \neq 0\}$.

In [M1, Definition 3.4] we introduced the following constant.

Definition 2.5. [M1] For $\beta = [\sum_{k=1}^n b_k D_k] \in A_{d-1}(\mathbf{P})$ and an ordered subset $I = \{i_0,\ldots,i_d\}$ of $\{1,\ldots,n\}$, let the constant c_I^β be the determinant of the $(d+1)\times(d+1)$ matrix obtained from $(\langle m_j, e_k\rangle_{1\leq j\leq d, k\in I})$ by adding the first row $(b_k)_{k\in I}$, where $m_1,\ldots,m_d$ is the same integer basis of the lattice M as in the definition of Ω.

The next result from [M1, Theorem 4.8] shows how to explicitly compute the toric residue map.

Theorem 2.6. [M1] *Let $\mathbf{P}$ be a complete toric variety, and let $\beta = [D] \in A_{d-1}(\mathbf{P})$ be big and nef. If $F_0,\ldots,F_d \in S_\beta$ do not vanish simultaneously on $\mathbf{P}$, then:*

(i) *the toric residue map $\mathrm{Res}_F : S_\rho/\langle F_0,\ldots,F_d\rangle_\rho \to \mathbb{C}$, $\rho = (d+1)\beta - \beta_0$, is an isomorphism,*

(ii) $\operatorname{Res}_F(J_F) = d!\operatorname{vol}(\Delta_D)$, where $J_F = \det(\partial F_j/\partial x_{i_k})/c_I^\beta \hat{x}_I \in S_{(d+1)\beta-\beta_0}$ is the toric Jacobian of $F_0, \ldots, F_d$, and $c_I^\beta \neq 0$.

When $X \subset \mathbf{P}$ is a nondegenerate hypersurface determined by $f \in S_\beta$, by Lemma 4.10 in [M1], the polynomials $F_k = x_k\partial f/\partial x_k$, $k \in I$, do not vanish simultaneously on $\mathbf{P}$ for $c_I^\beta \neq 0$, and $J_0(f) = \langle x_k\partial f/\partial x_k : k \in I\rangle$. Therefore, there is the corresponding toric residue map $\operatorname{Res}_{F_I} : S_\rho/J_0(f)_\rho \to \mathbb{C}$. However, the diagram (10) in [M1] tells us that this map depends on the choice of I, while $c_I^\beta\operatorname{Res}_{F_I}$ doesn't. By the above theorem, $c_I^\beta\operatorname{Res}_{F_I}(J) = d!\operatorname{vol}(\Delta_D)$ for

$$J = \frac{\det(\partial F_j/\partial x_k)_{k,j\in I}}{(c_I^\beta)^2\hat{x}_I}. \tag{6}$$

For an i-semiample nondegenerate hypersurface $X \subset \mathbf{P}_\Sigma$ and $\sigma \in \Sigma_X$. By Proposition 2.2, there is an isomorphism

$$R_0^\sigma(f)_{(i-\dim\sigma+1)\beta-\beta_0+\beta_1^\sigma} \cong R_0(f_{\sigma'})_{(i-\dim\sigma+1)\beta^{\sigma'}-\beta_0^{\sigma'}} \cong R_0(f_\sigma)_{(i-\dim\sigma+1)\beta^\sigma-\beta_0^\sigma},$$

composition of which we denote by $\pi_{\sigma*}$. Since $Y \cap V(\sigma)$ is an ample hypersurface in $V(\sigma)$, determined by f_σ, we get the corresponding map

$$c_I^{\beta^\sigma}\operatorname{Res}_{F_{\sigma I}} : R_0(f_\sigma)_{(i-\dim\sigma+1)\beta^\sigma-\beta_0^\sigma} \cong \mathbb{C}.$$

It is not difficult to find a polynomial in $R_0(f_\sigma)_{(i-\dim\sigma+1)\beta^\sigma-\beta_0^\sigma}$, which $\varphi_{\sigma'*}$ and $\pi_{\sigma*}$ map to the polynomials (6) associated with $f_{\sigma'} \in S(V(\sigma'))_{\beta^{\sigma'}}$ and $f_\sigma \in S(V(\sigma))_{\beta^\sigma}$, respectively. This polynomial is

$$J_\sigma = \frac{\det(x_k\partial F_j/\partial x_k)_{k,j\in I}}{(c_I^{\beta,\sigma})^2\prod_{\tilde{\pi}(\rho_k)\not\subset\sigma} x_k},$$

where $c_I^{\beta,\sigma} \neq 0$ for an ordered subset I of $\{1,\ldots,n\}$ is the determinant of the $(i(\sigma)+1) \times (i(\sigma)+1)$ matrix obtained from $(\langle m_j^\sigma, e_k\rangle)_{1\leq j\leq i(\sigma), k\in I}$ by adding the first row $(b_i)_{i\in I}$ such that $\beta = [\sum_{k=1}^n b_kD_k]$ with $b_k = 0$ if $\rho_k \subset \sigma$, and $m_1^\sigma, \ldots, m_{i(\sigma)}^\sigma$ is the same integer basis of the lattice $M_X \cap \sigma^\perp$ as in the definition of Ω_σ. One can verify that J_σ does not depend on the choice of I, and the fact that J_σ is indeed a polynomial follows from the arguments at the end of the proof of [C, Proposition 4.1]. Using Theorem 2.6, we conclude that

$$c_I^{\beta^\sigma}\operatorname{Res}_{F_{\sigma I}}\pi_{\sigma*}(J_\sigma) = (i-\dim\sigma)!\operatorname{vol}(\Gamma_\sigma), \tag{7}$$

where Γ_σ is the face of Δ_D corresponding to $\sigma \in \Sigma_X$.

In the case $\dim\sigma = i-1$, we will also need to use a special map constructed the following way. By equation (7),

$$C \prod_{\tilde{\pi}(\rho_k)\not\subset\sigma} x_k - \frac{c_I^{\beta^\sigma}\operatorname{Res}_{F_{\sigma I}}\pi_{\sigma*}(C\prod_{\tilde{\pi}(\rho_k)\not\subset\sigma} x_k)}{\operatorname{vol}(\Gamma_\sigma)}J_\sigma \in J_0^\sigma(f)_{2\beta-\beta_0+\beta_1^\sigma} \tag{8}$$

for $C \in (S/\langle x_k : \tilde{\pi}(\rho_k) \subset \sigma\rangle)_{2\beta-2\beta_0+2\beta_1^\sigma}$. The ideal $J_0^\sigma(f)$ is generated by $J_0(f)$ and x_k for $\tilde{\pi}(\rho_k) \subset \sigma$, while the ideal $J_0(f)$ has two minimal generators f and x_sf_s if $\tilde{\pi}(\rho_s) \not\subset \sigma$. Define a projection p_σ as the composition of the maps:

$$(S/\langle x_k : \tilde{\pi}(\rho_k) \subset \sigma\rangle)_{2\beta-2\beta_0+2\beta_1^\sigma} \to J_0^\sigma(f)_\rho \to J_0^\sigma(f)_\rho/\langle f, x_k : \tilde{\pi}(\rho_k) \subset \sigma\rangle_\rho$$

$$\cong (\langle x_sf_s\rangle/\langle x_k : \tilde{\pi}(\rho_k) \subset \sigma\rangle)_\rho \cong (S/\langle x_k : \tilde{\pi}(\rho_k) \subset \sigma\rangle)_{\beta-\beta_0+\beta_1^\sigma} = R_1^\sigma(f)_{\beta-\beta_0+\beta_1^\sigma},$$

where $\rho = 2\beta - \beta_0 + \beta_1^\sigma$, the first map is defined by (8) and the fourth map is a division by $x_s f_s/\langle m_1^\sigma, e_s\rangle$. The map p_σ is independent of the choice of s. Indeed, since

$$\frac{e_{s_1}}{\langle m_1^\sigma, e_{s_1}\rangle} - \frac{e_{s_2}}{\langle m_1^\sigma, e_{s_2}\rangle}$$

belongs to the space spanned by e_k for $\tilde{\pi}(\rho_k) \subset \sigma$, there is an Euler formula (see [BC, Definition 3.9]) showing that

$$\frac{x_{s_1} f_{s_1}}{\langle m_1^\sigma, e_{s_1}\rangle} - \frac{x_{s_2} f_{s_2}}{\langle m_1^\sigma, e_{s_2}\rangle}$$

belongs to the ideal $\langle f, x_k : \tilde{\pi}(\rho_k) \subset \sigma\rangle$.

For $\sigma \in \Sigma_X$, define $\mathrm{Res}^\sigma : R_1^\sigma(f) \to \mathbb{C}$ by 0 in all degrees except $i(\sigma) + 1)\beta - 2\beta_0 + 2\beta_1^\sigma$, for which Res^σ is the composition

$$R_1^\sigma(f)_{(i(\sigma)+1)\beta-2\beta_0+2\beta_1^\sigma} \longrightarrow R_0^\sigma(f)_{(i(\sigma)+1)\beta-\beta_0+\beta_1^\sigma} \xrightarrow{c_I^{\beta^\sigma} \mathrm{Res}_{F_\sigma I} \pi_{\sigma*}} \mathbb{C},$$

where the first arrow is a multiplication by $\prod_{\tilde{\pi}(\rho_k) \not\subset \sigma} x_k/((i - \dim\sigma)! \, \mathrm{vol}(\Gamma_\sigma))$.

Now, we are ready to show how to partially split the residue part $H_{\mathrm{res}}^s(X \cap V(\tau))$.

Proposition 2.7. *Let $X \subset \mathbf{P}_\Sigma$ be an i-semiample nondegenerate hypersurface defined by $f \in S_\beta$. Given $\tau \in \Sigma$ and the smallest cone $\sigma \in \Sigma_X$, containing the image of τ, then:* (i) *for $p + q \neq i - \dim\sigma - 1$,*

$$H_{\mathrm{res}}^{p,q}(X \cap V(\tau)) = \sum_{\tau \subset \tau' \in \Sigma(\dim\tau+1)} \varphi_{\tau,\tau'!} H_{\mathrm{res}}^{p-1,q-1}(X \cap V(\tau')),$$

and

$$H_{\mathrm{res}}^{p,q}(X \cap V(\tau)) \cong R_1^\sigma(f)_{(q+1)\beta-\beta_0+\beta_1^\sigma} \bigoplus_{\substack{\tau \subset \tau' \\ \tau' \in \Sigma(\dim\tau+1)}} \varphi_{\tau,\tau'!} H_{\mathrm{res}}^{p-1,q-1}(X \cap V(\tau'))$$

for $p + q = i - \dim\sigma - 1$, where the first factor is included by the residue map, (ii) *for $\dim\sigma < i - 1$, $A \in R_1^\sigma(f)_{(p+1)\beta-\beta_0+\beta_1^\sigma}$, $B \in R_1^\sigma(f)_{(q+1)\beta-\beta_0+\beta_1^\sigma}$,*

$$\mathrm{Res}_\tau(A) \cup \mathrm{Res}_\tau(B) = c_p^\sigma \mathrm{Res}^\sigma(AB)\varphi_\tau^* i^* X^{i(\sigma)-1},$$

where

$$c_p^\sigma = -(-2\pi\sqrt{-1})^{i(\sigma)-1}\frac{(-1)^{(i(\sigma)-1)(i(\sigma)+2p+2)}}{p!(i(\sigma)-p-1)!},$$

if $p + q = i - \dim\sigma - 1$, and

$$\mathrm{Res}_\tau(A) \cup \mathrm{Res}_\tau(B) = 0$$

if $p + q \neq i - \dim\sigma - 1$, and, for $\dim\sigma = i - 1$,

$$\mathrm{Res}_\tau(A) \cup \mathrm{Res}_\tau(B) = -\mathrm{Res}^\sigma(AB)\varphi_\tau^* i^*[\mathbf{P}_\Sigma] + \mathrm{Res}_\tau(p_\sigma(AB)).$$

Proof. (i) From the exact sequence (1) and the vanishing

$$\mathrm{Gr}_s^W H^s(X \cap \mathbf{T}_\tau) \cong \mathrm{Gr}_s^W H^s((Y \cap \mathbf{T}_\sigma) \times (\mathbb{C}^*)^{d-i+\dim\sigma-\dim\tau}) = 0$$

for $s \neq i - \dim\sigma - 1$, by equations (2) and (11) in [M2], we conclude the first part of the statement.

For the second part, we first use an induction on the dimension of $\tau \in \Sigma$ such that $\sigma \in \Sigma_X$ is the smallest cone containing the image of τ to show that

$$\mathrm{Res}_\tau(J_1^\sigma(f)_{(q+1)\beta-\beta_0+\beta_1^\sigma}) = 0 \tag{9}$$

in the cohomology of $X \cap V(\tau)$. By Proposition 2.2 and [M2, Theorem 4.4], this is true for $\tau \in \Sigma(d - i + \dim \sigma)$ such that $\tilde{\pi}(\tau) \subset \sigma$, because $X \cap V(\tau)$ in $V(\tau)$ is big and nef. By induction, assume that (9) holds for $\tau' \in \Sigma(k)$ such that $\sigma \in \Sigma_X$ is the smallest cone containing $\tilde{\pi}(\tau')$. We need to prove (9) for $\tau \in \Sigma(k-1)$ satisfying the same condition. Let $A \in J_1^\sigma(f)_{(q+1)\beta - \beta_0 + \beta_1^\sigma}$, then, by Poincaré duality, $\mathrm{Res}_\tau(A) = 0$ in $H^{p,q}(X \cap V(\tau))$, $p = i - \dim \sigma - 1 - q$, if and only if $\int_X$ of the cup product of $\mathrm{Res}_\tau(A)$ with all elements in $H^{d(\tau)-1-p,d(\tau)-1-q}(X \cap V(\tau))$ vanishes (note that X is not necessarily connected). From Section 1 we know that

$$H^*(X \cap V(\tau)) = H^*_{\mathrm{toric}}(X \cap V(\tau)) \oplus H^*_{\mathrm{res}}(X \cap V(\tau)),$$

and

$$\int_X H^*_{\mathrm{toric}}(X \cap V(\tau)) \cup H^*_{\mathrm{res}}(X \cap V(\tau)) = 0. \tag{10}$$

On the other hand, for $\dim \tau \neq d - i + \dim \sigma$,

$$H^{d(\tau)-1-p,d(\tau)-1-q}_{\mathrm{res}}(X \cap V(\tau)) = \sum_{\tau \subset \tau' \in \Sigma(\dim \tau + 1)} \varphi_{\tau,\tau'!} H^{d(\tau)-2-p,d(\tau)-2-q}_{\mathrm{res}}(X \cap V(\tau'))$$

by the first part of this proposition. Since

$$\mathrm{Res}_\tau(A) \cup \varphi_{\tau,\tau'!} h = \varphi_{\tau,\tau'!}(\varphi^*_{\tau,\tau'} \mathrm{Res}_\tau(A) \cup h)$$

by the Gysin projection formula, it is enough to show $\varphi^*_{\tau,\tau'}(\mathrm{Res}_\tau(A)) = 0$ for $\tau \subset \tau' \in \Sigma(\dim \tau + 1)$. If $\tilde{\pi}(\tau') \not\subset \sigma$, then there is $\rho_l \subset \tau'$ such that $\tilde{\pi}(\rho_l) \not\subset \sigma$. In this case, by Proposition 2.3, $\varphi^*_{\tau,\tau'} \mathrm{Res}_\tau(A)$ is represented by the restriction of the Čech cocycle

$$\frac{(-1)^{i(\sigma)-1+(q(q+1)/2)}}{q!} \left\{ \frac{A K_{i_q} \cdots K_{i_0} \Omega_\sigma}{f_{i_0} \cdots f_{i_q}} \right\}_{i_0 \ldots i_q}$$

to $X \cap V(\tau')$. But each term in the form $K_{i_q} \cdots K_{i_0} \Omega_\sigma$ on a nonempty open set $U_{i_q} \cap \cdots \cap U_{i_0} \cap V(\tau')$ contains $\mathrm{d}x_l$ or x_l and, therefore, vanishes on $X \cap V(\tau')$. If $\tilde{\pi}(\tau') \subset \sigma$, then, by the induction assumption, $\mathrm{Res}_{\tau'}(A) = 0$ in $H^{p,q}(X \cap V(\tau'))$. The commutative diagram

$$
\begin{array}{ccc}
S_{(q+1)\beta - \beta_0 + \beta_1^\sigma} & \xrightarrow{\mathrm{Res}_\tau} & H^q(X \cap V(\tau), \Omega^{i(\sigma)-1-q}_{X \cap V(\tau)}) \\
\| & & \downarrow{\varphi^*_{\tau,\tau'}} \\
S_{(q+1)\beta - \beta_0 + \beta_1^\sigma} & \xrightarrow{\mathrm{Res}_{\tau'}} & H^q(X \cap V(\tau'), \Omega^{i(\sigma)-1-q}_{X \cap V(\tau')}).
\end{array}
$$

gives $\varphi^*_{\tau,\tau'} \mathrm{Res}_\tau(A) = 0$ again. This finishes the proof of (9) by induction, which shows that the map

$$R_1^\sigma(f)_{(q+1)\beta - \beta_0 + \beta_1^\sigma} \xrightarrow{\mathrm{Res}_\tau} H^{p,q}_{\mathrm{res}}(X \cap V(\tau))$$

is well defined.

Finally, for $\sigma' \in \Sigma(d - i + \dim \sigma)$ such that $\tilde{\pi}(\sigma') \subset \sigma$, we have a diagram

$$
\begin{array}{ccccc}
R_1^\sigma(f)_{(q+1)\beta - \beta_0 + \beta_1^\sigma} & \xrightarrow{\mathrm{Res}_\tau} & H^{p,q}_{\mathrm{res}}(X \cap V(\tau)) & \longrightarrow & H^{p,q}\big(PH^{p+q}(X \cap \mathbf{T}_\tau)\big) \\
\downarrow & & \downarrow & & \\
R_1(f_{\sigma'})_{(q+1)\beta^{\sigma'} - \beta_0^{\sigma'}} & \xrightarrow{\mathrm{Res}_{\sigma'}} & H^{p,q}_{\mathrm{res}}(X \cap V(\sigma')) & \longrightarrow & H^{p,q}\big(PH^{p+q}(X \cap \mathbf{T}_{\sigma'})\big)
\end{array}
\tag{11}
$$

and, by equation (12) in [M2], the isomorphisms

$$H^{p,q}\big(PH^{p+q}(X \cap \mathbf{T}_\tau)\big) \cong H^{p,q}\big(PH^{p+q}(\pi(X) \cap \mathbf{T}_\sigma)\big) \cong H^{p,q}\big(PH^{p+q}(X \cap \mathbf{T}_{\sigma'})\big)$$

induced by $\pi : \mathbf{P}_\Sigma \to \mathbf{P}_{\Sigma_X}$. Since $X \cap V(\sigma')$ in $V(\sigma')$ is big and nef, the composition on the bottom of (11) is an isomorphism, by Theorem 4.4 in [M1]. Hence, the same holds for the composition on the top of (11). Now, the exact sequence (1) produces the result.

(ii) Let $\dim \sigma < i - 1$. As in part (i), we prove the formula

$$\mathrm{Res}_\tau(A) \cup \mathrm{Res}_\tau(B) = c_p^\sigma \mathrm{Res}^\sigma(AB) \varphi_\tau^* i^* X^{i(\sigma)-1} \qquad (12)$$

by an induction on the dimension of $\tau \in \Sigma$ such that $\sigma \in \Sigma_X$ is the smallest cone containing the image of τ. When $\dim \tau = d - i + \dim \sigma$, the hypersurface $X \cap V(\tau)$ in $V(\tau)$ is big and nef. Therefore, by the results in [M1, page 104] and the definition of Res^σ,

$$\int_{X \cap V(\tau)} \mathrm{Res}_\tau(A) \cup \mathrm{Res}_\tau(B) = c_p^\sigma (i - \dim \sigma)! \mathrm{vol}(\Gamma_\sigma) \mathrm{Res}^\sigma(AB)$$

(the formula we use in [M1, page 104] must be corrected by a factor of $2\pi\sqrt{-1}$). By [F, Section 5.3], $(X \cap V(\tau))^{i(\sigma)} = (i - \dim \sigma)! \mathrm{vol}(\Gamma_\sigma)$, whence (12) holds in this case. Assume that (12) holds for $\dim \tau = k$, we will prove (12) for $\dim \tau = k - 1$. By Poincaré duality, it suffices to show that the cup product of both sides of (12) with an element of $H^{d(\tau)-i(\sigma),d(\tau)-i(\sigma)}(X \cap V(\tau))$ produces the same result. The cohomology of $X \cap V(\tau)$ decomposes into the toric and residue parts. Using the relations in $H^*_{\mathrm{toric}}(X \cap V(\tau))$, coming from the cohomology of the ambient toric variety, we can assume that an element from the toric part is $\varphi_\tau^* i^* [V(\gamma)]$ with $\gamma \in \Sigma(\dim \sigma - \dim \tau)$ such that $\gamma \cap \tau = 0$. Then we have

$$\varphi_\tau^* i^* [V(\gamma)] = \alpha_{\gamma\tau}[X \cap V(\tau + \gamma)],$$

where $\alpha_{\gamma\tau} = \frac{\mathrm{mult}(\gamma)\mathrm{mult}(\tau)}{\mathrm{mult}(\tau+\gamma)}$ in $H^*(X \cap V(\tau))$, because

$$[V(\tau)] \cdot [V(\gamma)] = \alpha_{\gamma\tau}[X \cap V(\tau + \gamma)]$$

in the Chow ring $A^*(\mathbf{P}_\Sigma)$. Hence,

$$\varphi_\tau^* i^* [V(\gamma)] \cup \mathrm{Res}_\tau(A) \cup \mathrm{Res}_\tau(B) = \alpha_{\gamma\tau} \varphi_{\tau+\gamma,\tau!}(\varphi_{\tau+\gamma,\tau}^* \mathrm{Res}_\tau(A) \cup \varphi_{\tau+\gamma,\tau}^* \mathrm{Res}_\tau(B))$$

and

$$\varphi_\tau^* i^* [V(\gamma)] \cup \varphi_\tau^* i^* X^{i(\sigma)-1} = \alpha_{\gamma\tau} \varphi_{\tau+\gamma,\tau!} \varphi_{\tau+\gamma,\tau}^* \varphi_\tau^* i^* X^{i(\sigma)-1}$$
$$= \alpha_{\gamma\tau} \varphi_{\tau+\gamma,\tau!} \varphi_{\tau+\gamma}^* i^* X^{i(\sigma)-1}.$$

We can assume that $\tau + \gamma$ forms a cone in Σ, and that γ lies in σ, because, otherwise, $\varphi_{\tau+\gamma,\tau}^* \mathrm{Res}_\tau(A) = 0$ by the same argument as in [M1, page 102], and $\varphi_{\tau+\gamma,\tau!} \varphi_{\tau+\gamma}^* i^* X^{i(\sigma)-1} = 0$ by Lemma 1.5. In this case,

$$\varphi_{\tau+\gamma,\tau}^* \mathrm{Res}_\tau(A) \cup \varphi_{\tau+\gamma,\tau}^* \mathrm{Res}_\tau(B) = \mathrm{Res}_{\tau+\gamma}(A) \cup \mathrm{Res}_{\tau+\gamma}(B)$$
$$= c_p^\sigma \mathrm{Res}^\sigma(AB) \varphi_{\tau+\gamma}^* i^* X^{i(\sigma)-1}$$

by the induction assumption. To finish the induction proof of (12), we also need to consider the elements $h \in H^*_{\mathrm{res}}(X \cap V(\tau))$, the cup product of which vanishes with the right hand side of (12), by equation (10). The same should hold for the

left hand side of (12). By part (i) of this proposition, there are two possibilities: $h = \varphi_{\tau,\tau'!}h'$ with $h' \in H^*_{\text{res}}(X \cap V(\tau'))$ or $h = \text{Res}_\tau(C)$. In the first case,

$$\varphi_{\tau,\tau'!}h' \cup \text{Res}_\tau(A) \cup \text{Res}_\tau(B) = \varphi_{\tau,\tau'!}(h' \cup \varphi^*_{\tau,\tau'}\text{Res}_\tau(A) \cup \varphi^*_{\tau,\tau'}\text{Res}_\tau(B)).$$

If $\tilde{\pi}(\tau') \not\subset \sigma$, then $\varphi^*_{\tau,\tau'}\text{Res}_\tau(A) = 0$ as above. If $\tilde{\pi}(\tau') \subset \sigma$, then we can apply the induction assumption to deduce that

$$\varphi^*_{\tau,\tau'}\text{Res}_\tau(A) \cup \varphi^*_{\tau,\tau'}\text{Res}_\tau(B) = \text{Res}_{\tau'}(A) \cup \text{Res}_{\tau'}(B)$$

belongs to $H^*_{\text{toric}}(X \cap V(\tau'))$, the cup product of which with h' is zero. In the second case, we have

$$\text{Res}_\tau(A) \cup \text{Res}_\tau(B) \cup \text{Res}_\tau(C) \in H^{d(\tau)-1,d(\tau)-1}(X \cap V(\tau))$$

where $d(\tau) = d - \dim\tau$. Up to a constant multiple, the cocycle that represents this cup product is

$$\left\{ ABC \frac{K_{i_p}\cdots K_{i_0}\Omega}{f_{i_0}\cdots f_{i_p}} \wedge \frac{K_{i_{i(\sigma)-1}}\cdots K_{i_p}\Omega}{f_{i_p}\cdots f_{i_{i(\sigma)-1}}} \wedge \frac{K_{i_{d(\tau)-1}}\cdots K_{i_{i(\sigma)-1}}\Omega}{f_{i_{i(\sigma)-1}}\cdots f_{i_{d(\tau)-1}}} \right\}_{i_0\ldots i_{d(\tau)-1}}$$

in $C^{d(\tau)-1}(\mathcal{U}|_{X\cap V(\tau)}, \Omega^{d(\tau)-1}_{X\cap V(\tau)})$. It follows from [C, Proposition 5.3] that there are $i - \dim\sigma$ open sets from the open cover $\mathcal{U}|_{X\cap V(\tau)}$ which cover $X \cap V(\tau)$. Passing to this refinement, we conclude that the above cup product vanishes, since $d - \dim\tau > i - \dim\sigma$. Thus, the equation (12) is shown. The proof of $\text{Res}_\tau(A) \cup \text{Res}_\tau(B) = 0$, for $p + q \neq i - \dim\sigma - 1$, follows the same steps as above.

When $\dim\sigma = i - 1$, we will use a different strategy. The cup product $\text{Res}_\tau(A) \cup \text{Res}_\tau(B)$ is represented by the Čech cocycle

$$\left\{ \frac{AB\langle m^\sigma_1, e_{i_0}\rangle^2 (\prod_{\tilde{\pi}(\rho_k)\not\subset\sigma} x_k)^2}{(x_{i_0}f_{i_0})^2} \right\}_{i_0}.$$

Write

$$AB \prod_{\tilde{\pi}(\rho_k)\not\subset\sigma} x_k = P + \text{Res}^\sigma(AB)J_\sigma,$$

where P is the polynomial determined by (8) with $C = AB$. Using the definition of the projection

$$p_\sigma : (S/\langle x_k : \tilde{\pi}(\rho_k) \subset \sigma\rangle)_{2\beta-2\beta_0+2\beta^\sigma_1} \longrightarrow R^\sigma_1(f)_{\beta-\beta_0+\beta^\sigma_1}$$

before this proposition, one can see that the cocycle

$$\left\{ \frac{P\langle m^\sigma_1, e_{i_0}\rangle^2 \prod_{\tilde{\pi}(\rho_k)\not\subset\sigma} x_k}{(x_{i_0}f_{i_0})^2} \right\}_{i_0}.$$

is equal to

$$\left\{ \frac{p_\sigma(AB)\langle m^\sigma_1, e_{i_0}\rangle \prod_{\tilde{\pi}(\rho_k)\not\subset\sigma} x_k}{x_{i_0}f_{i_0}} \right\}_{i_0},$$

because the difference is by multiples of f and x_k, such that $\tilde{\pi}(\text{int}\rho_k) \subset \text{int}\sigma$ and $\rho_k \subset \tau$, which vanish on $X \cap V(\tau)$. By Proposition 2.3, this cocycle represents $\text{Res}_\tau(p_\sigma(AB))$. Therefore, $\text{Res}_\tau(A) \cup \text{Res}_\tau(B) - \text{Res}_\tau(p_\sigma(AB))$ is represented by the cocycle

$$\left\{ \frac{\text{Res}^\sigma(AB)J_\sigma\langle m^\sigma_1, e_{i_0}\rangle^2 \prod_{\tilde{\pi}(\rho_k)\not\subset\sigma} x_k}{(x_{i_0}f_{i_0})^2} \right\}_{i_0}. \tag{13}$$

As above, we can change J_σ by multiples of f and x_k, such that $\tilde\pi(\mathrm{int}\rho_k) \subset \mathrm{int}\sigma$ and $\rho_k \subset \tau$, because they vanish on $X \cap V(\tau)$. Also, we can assume that the index i_0 takes values in $\{s_1, s_2\}$ for $c_{s_1 s_2}^{\beta,\sigma} \neq 0$, since $x_{s_1} f_{s_1}$ and $x_{s_2} f_{s_2}$ do not vanish simultaneously on $V(\tau)$, by [M1, Lemma 4.10]. Using an Euler formula, similar to the one in the proof of Proposition 4.7 in [M1], we get that

$$c_{s_1 s_2}^{\beta,\sigma} f - \langle m_1^\sigma, e_{s_2}\rangle x_{s_1} f_{s_1} + \langle m_1^\sigma, e_{s_1}\rangle x_{s_2} f_{s_2}$$

with $c_{s_1 s_2}^{\beta,\sigma} \neq 0$ is a multiple of x_k, such that $\tilde\pi(\mathrm{int}\rho_k) \subset \mathrm{int}\sigma$ and $\rho_k \subset \tau$. Hence, modulo these multiples and multiples of f,

$$J_\sigma = \frac{\det\begin{pmatrix} (x_{s_1}\partial_{s_1})^2 f & (x_{s_1}\partial_{s_1})(x_{s_2}\partial_{s_2})f \\ (x_{s_2}\partial_{s_2})(x_{s_1}\partial_{s_1})f & (x_{s_2}\partial_{s_2})^2 f \end{pmatrix}}{(c_{s_1 s_2}^{\beta,\sigma})^2 \prod_{\tilde\pi(\rho_k)\not\subset\sigma} x_k}$$

$$\equiv \frac{\det\begin{pmatrix} x_{s_1} f_{s_1} & x_{s_2} f_{s_2} \\ x_{s_1} x_{s_2} f_{s_1 s_2} & x_{s_2} f_{s_2} + x_{s_2}^2 f_{s_2 s_2} \end{pmatrix}}{\langle m_1^\sigma, e_{s_2}\rangle c_{s_1 s_2}^{\beta,\sigma} \prod_{\tilde\pi(\rho_k)\not\subset\sigma} x_k} \equiv \frac{\det\begin{pmatrix} f & x_{s_2} f_{s_2} \\ x_{s_2} f_{s_2} & x_{s_2} f_{s_2} + x_{s_2}^2 f_{s_2 s_2} \end{pmatrix}}{\langle m_1^\sigma, e_{s_2}\rangle^2 \prod_{\tilde\pi(\rho_k)\not\subset\sigma} x_k}$$

$$\equiv -\frac{(x_{s_2} f_{s_2})^2}{\langle m_1^\sigma, e_{s_2}\rangle^2 \prod_{\tilde\pi(\rho_k)\not\subset\sigma} x_k} \equiv -\frac{(x_{s_1} f_{s_1})^2}{\langle m_1^\sigma, e_{s_1}\rangle^2 \prod_{\tilde\pi(\rho_k)\not\subset\sigma} x_k}.$$

Therefore, the cocycle (13) coincides with $\{-\mathrm{Res}^\sigma(AB)\}_{i_0}$, which comes from the restriction of the global constant function $-\mathrm{Res}^\sigma(AB) \in H^0(V(\tau), \mathcal{O}_{V(\tau)}) \cong \mathbb{C}$. We have a commutative diagram:

$$\begin{array}{ccc} H^0(X \cap V(\tau)) & \cong & H^0(X \cap V(\tau), \mathcal{O}_{X\cap V(\tau)}) \\ i^* \uparrow & & i^* \uparrow \\ H^0(V(\tau)) & \cong & H^0(V(\tau), \mathcal{O}_{V(\tau)}). \end{array}$$

The isomorphism on the bottom of this diagram sends the fundamental class $[V(\tau)]$ to 1, whence the isomorphism on the top sends $-\mathrm{Res}^\sigma(AB)i^*[V(\tau)]$ to the element represented by the cocycle (13). This finishes the proof of the proposition. $\qquad\square$

After all the intermediate technical steps we can describe the generators which span the residue part of the cohomology of semiample nondegenerate hypersurfaces.

Proposition 2.8. *Let $X \subset \mathbf{P}_\Sigma$ be an i-semiample nondegenerate hypersurface defined by $f \in S_\beta$. Then*

$$H_{\mathrm{res}}^{p,q}(X) \cong \sum_{\substack{\sigma \in \Sigma_X \\ \gamma \in \Sigma((\dim\sigma + p + q + 1 - i)/2) \\ \tilde\pi(\mathrm{int}\gamma)\subset\mathrm{int}\sigma}} \varphi_{\gamma!}\mathrm{Res}_\gamma R_1^\sigma(f)_{(q+1-\dim\gamma)\beta - \beta_0 + \beta_1^\sigma}.$$

Proof. We can actually prove by induction on the codimension of $\tau \in \Sigma$ a more general statement:

$$H_{\mathrm{res}}^{p,q}(X \cap V(\tau)) \cong \sum_{\substack{\tilde\pi(\tau)\subset\sigma\in\Sigma_X \\ \tau\subset\gamma\in\Sigma((p+q+1-i(\sigma))/2+\dim\tau) \\ \tilde\pi(\mathrm{int}\gamma)\subset\mathrm{int}\sigma}} \varphi_{\tau,\gamma!}\mathrm{Res}_\gamma R_1^\sigma(f)_{(q+1-\dim\gamma+\dim\tau)\beta - \beta_0 + \beta_1^\sigma}.$$

This is certainly true for the maximally dimensional $\tau \in \Sigma(d)$, because $X \cap V(\tau)$ is empty by Proposition 1.6 and Remark 2.2 in [M2] and there is no γ satisfying the conditions on the right side of the formula. Assume that the formula holds for

$\tau \in \Sigma(k-1)$. Then Proposition 2.7 and the induction assumption produce the result for $\tau \in \Sigma(k)$. $\qquad\square$

The next result describes the product structure on the residue part of the cohomology of semiample nondegenerate hypersurfaces.

Proposition 2.9. *Let $X \subset \mathbf{P}_\Sigma$ be an i-semiample nondegenerate hypersurface defined by $f \in S_\beta$, then:*

(i) for $\sigma \in \Sigma_X$ with $\dim\sigma < i-1$, $\gamma_1, \gamma_2 \in \Sigma$, such that $\tilde\pi(\mathrm{int}\gamma_k) \subset \mathrm{int}\sigma$ with $k = 1, 2$, and $A \in R_1^\sigma(f)_{(p+1)\beta - \beta_0 + \beta_1^\sigma}$, $B \in R_1^\sigma(f)_{*\beta - \beta_0 + \beta_1^\sigma}$,

$$\frac{\varphi_{\gamma_1!}\mathrm{Res}_{\gamma_1}(A)}{\mathrm{mult}(\gamma_1)} \cup \frac{\varphi_{\gamma_2!}\mathrm{Res}_{\gamma_2}(B)}{\mathrm{mult}(\gamma_2)} = c_p^\sigma \mathrm{Res}^\sigma(AB) i^* X^{i(\sigma)-1} \prod_{\rho_k \subset \gamma_1} i^* D_k \prod_{\rho_k \subset \gamma_2} i^* D_k,$$

where $i(\sigma) = i - \dim\sigma$ and

$$c_p^\sigma = -(-2\pi\sqrt{-1})^{i(\sigma)-1} \frac{(-1)^{(i(\sigma)-1)(i(\sigma)+2p+2)}}{p!(i(\sigma)-p-1)!},$$

(ii) for $\dim\sigma = i-1$, A, B and γ_1, γ_2, as in part (i),

$$\frac{\varphi_{\gamma_1!}\mathrm{Res}_{\gamma_1}(A)}{\mathrm{mult}(\gamma_1)} \cup \frac{\varphi_{\gamma_2!}\mathrm{Res}_{\gamma_2}(B)}{\mathrm{mult}(\gamma_2)} = -\mathrm{Res}^\sigma(AB) i^*[\mathbf{P}_\Sigma] \prod_{\rho_k \subset \gamma_1} i^* D_k \prod_{\rho_k \subset \gamma_2} i^* D_k$$

$$+ \prod_{\rho_k \subset \gamma_1 \cap \gamma_2} i^* D_k \frac{\varphi_{\gamma_1+\gamma_2!}\mathrm{Res}_{\gamma_1+\gamma_2}(p_\sigma(AB))}{\mathrm{mult}(\gamma_1 + \gamma_2)},$$

(iii) for $\gamma_1, \gamma_2 \in \Sigma$ satisfying neither part (i) nor (ii),

$$\varphi_{\gamma_1!}\mathrm{Res}_{\gamma_1}(A) \cup \varphi_{\gamma_2!}\mathrm{Res}_{\gamma_2}(B) = 0.$$

Proof. Applying the Gysin projection formula, we get

$\varphi_{\gamma_1!}\mathrm{Res}_{\gamma_1}(A) \cup \varphi_{\gamma_2!}\mathrm{Res}_{\gamma_2}(B)$

$\qquad = \varphi_{\gamma_1\cap\gamma_2!}\varphi_{\gamma_1\cap\gamma_2,\gamma_1!}\mathrm{Res}_{\gamma_1}(A) \cup \varphi_{\gamma_1\cap\gamma_2!}\varphi_{\gamma_1\cap\gamma_2,\gamma_2!}\mathrm{Res}_{\gamma_2}(B)$

$\qquad = \varphi_{\gamma_1\cap\gamma_2!}(\varphi^*_{\gamma_1\cap\gamma_2}\varphi_{\gamma_1\cap\gamma_2!}\varphi_{\gamma_1\cap\gamma_2,\gamma_1!}\mathrm{Res}_{\gamma_1}(A) \cup \varphi_{\gamma_1\cap\gamma_2,\gamma_2!}\mathrm{Res}_{\gamma_2}(B))$

$\qquad = \varphi_{\gamma_1\cap\gamma_2!}\varphi^*_{\gamma_1\cap\gamma_2}\varphi_{\gamma_1\cap\gamma_2!}(\varphi_{\gamma_1\cap\gamma_2,\gamma_1!}\mathrm{Res}_{\gamma_1}(A) \cup \varphi_{\gamma_1\cap\gamma_2,\gamma_2!}\mathrm{Res}_{\gamma_2}(B))$

$\qquad = \varphi_{\gamma_1\cap\gamma_2!}\varphi^*_{\gamma_1\cap\gamma_2}\varphi_{\gamma_1\cap\gamma_2!}\varphi_{\gamma_1\cap\gamma_2,\gamma_2!}(\varphi^*_{\gamma_1\cap\gamma_2,\gamma_2}\varphi_{\gamma_1\cap\gamma_2,\gamma_1!}\mathrm{Res}_{\gamma_1}(A) \cup \mathrm{Res}_{\gamma_2}(B))$

$\qquad = \varphi_{\gamma_1\cap\gamma_2!}\varphi^*_{\gamma_1\cap\gamma_2}\varphi_{\gamma_2!}(\alpha_{\gamma_1\gamma_2}\varphi_{\gamma_2,\gamma_1+\gamma_2!}\varphi^*_{\gamma_1,\gamma_1+\gamma_2}\mathrm{Res}_{\gamma_1}(A) \cup \mathrm{Res}_{\gamma_2}(B))$

$\qquad = \alpha_{\gamma_1\gamma_2}\varphi_{\gamma_1\cap\gamma_2!}\varphi^*_{\gamma_1\cap\gamma_2}\varphi_{\gamma_2!}\varphi_{\gamma_2,\gamma_1+\gamma_2!}(\varphi^*_{\gamma_1,\gamma_1+\gamma_2}\mathrm{Res}_{\gamma_1}(A) \cup \varphi^*_{\gamma_2,\gamma_1+\gamma_2}\mathrm{Res}_{\gamma_2}(B))$

$\qquad = \alpha_{\gamma_1\gamma_2}\varphi_{\gamma_1\cap\gamma_2!}\varphi^*_{\gamma_1\cap\gamma_2}\varphi_{\gamma_1+\gamma_2!}(\varphi^*_{\gamma_1,\gamma_1+\gamma_2}\mathrm{Res}_{\gamma_1}(A) \cup \varphi^*_{\gamma_2,\gamma_1+\gamma_2}\mathrm{Res}_{\gamma_2}(B))$

$\qquad = \alpha_{\gamma_1\gamma_2} i^*[V(\gamma_1 \cap \gamma_2)]\varphi_{\gamma_1+\gamma_2!}(\varphi^*_{\gamma_1,\gamma_1+\gamma_2}\mathrm{Res}_{\gamma_1}(A) \cup \varphi^*_{\gamma_2,\gamma_1+\gamma_2}\mathrm{Res}_{\gamma_2}(B)),$

where we also used the commutative diagram (see [M1, Lemma 5.4]):

$$
\begin{array}{ccc}
H^*(X \cap V(\gamma_1)) & \xrightarrow{\;\varphi_{\gamma_1\cap\gamma_2,\gamma_1!}\;} & H^*(X \cap V(\gamma_1 \cap \gamma_2)) \\
{\scriptstyle\varphi^*_{\gamma_1,\gamma_1+\gamma_2}}\Big\downarrow & & {\scriptstyle\varphi^*_{\gamma_1\cap\gamma_2}}\Big\downarrow \\
H^*(X \cap V(\gamma_1 + \gamma_2)) & \xrightarrow{\;\alpha_{\gamma_1\gamma_2}\varphi_{\gamma_2,\gamma_1+\gamma_2!}\;} & H^*(X \cap V(\gamma_2)),
\end{array}
$$

with $\alpha_{\gamma_1\gamma_2}$ satisfying the equality $[X\cap V(\gamma_1)] \cup [X\cap V(\gamma_2)] = \alpha_{\gamma_1\gamma_2}[X\cap V(\gamma_1+\gamma_2)]$ in $H^*(X \cap V(\gamma_1 \cap \gamma_2))$.

Note that the cup product calculated above is zero, if γ_1 and γ_2 do not span a cone in Σ. It also vanishes when γ_1 and γ_2 have different smallest cones σ_1 and

σ_2 in Σ_X containing them. Indeed, we can assume that σ_1 is not contained in σ_2. Then $\varphi^*_{\gamma_1, \gamma_1 + \gamma_2} \mathrm{Res}_{\gamma_1}(A)$ vanishes by the same argument as in [M1, p. 102].

To determine the constant $\alpha_{\gamma_1 \gamma_2}$ first notice that $[X \cap V(\gamma)] = i^*[V(\gamma)]$, for $\gamma \in \Sigma$, by the arguments in the proof of Lemma 5.7 in [M1]. On the other hand, we can write $\gamma_k = \gamma'_k + \gamma_1 \cap \gamma_2$, for $k = 1, 2$, so that $\gamma' \in \Sigma$ and $\dim \gamma_k = \dim \gamma'_k + \dim(\gamma_1 \cap \gamma_2)$. In this case,

$$[V(\gamma)] = \frac{\mathrm{mult}(\gamma)}{\mathrm{mult}(\gamma')\mathrm{mult}(\gamma_1 \cap \gamma_2)} \varphi^*_{\gamma_1 \cap \gamma_2}[V(\gamma')]$$

in $H^*(V(\gamma_1 \cap \gamma_2))$, where γ is γ_1, γ_2 or $\gamma_1 + \gamma_2$ and γ' is γ'_1, γ'_2 or $\gamma'_1 + \gamma'_2$, respectively. This is because

$$V(\gamma) = \frac{\mathrm{mult}(\gamma)}{\mathrm{mult}(\gamma')\mathrm{mult}(\gamma_1 \cap \gamma_2)} V(\gamma_1 \cap \gamma_2) \cdot V(\gamma')$$

in the Chow ring $A^*(\mathbf{P}_\Sigma)$ for the corresponding γ and γ'. Since

$$V(\gamma'_1) \cdot V(\gamma'_2) = \frac{\mathrm{mult}(\gamma'_1)\mathrm{mult}(\gamma'_2)}{\mathrm{mult}(\gamma'_1 + \gamma'_2)} V(\gamma'_1 + \gamma'_2),$$

it is not difficult to calculate

$$\alpha_{\gamma_1 \gamma_2} = \frac{\mathrm{mult}(\gamma_1)\mathrm{mult}(\gamma_2)}{\mathrm{mult}(\gamma_1 \cap \gamma_2)\mathrm{mult}(\gamma_1 + \gamma_2)}.$$

Hence,

$$\varphi_{\gamma_1!}\mathrm{Res}_{\gamma_1}(A) \cup \varphi_{\gamma_2!}\mathrm{Res}_{\gamma_2}(B)$$
$$= \frac{\mathrm{mult}(\gamma_1)\mathrm{mult}(\gamma_2)}{\mathrm{mult}(\gamma_1 + \gamma_2)} \prod_{\rho_k \subset \gamma_1 \cap \gamma_2} i^* D_k \varphi_{\gamma_1 + \gamma_2!}(\mathrm{Res}_{\gamma_1 + \gamma_2}(A) \cup \mathrm{Res}_{\gamma_1 + \gamma_2}(B))$$

Applying Proposition 2.8, we get the result. $\qquad\square$

To give a very explicit description of the cohomology of semiample hypersurfaces we introduce the following rings.

Definition 2.10. Given a semiample class $[D] \in A_{d-1}(\mathbf{P}_\Sigma)$ and $\sigma \in \Sigma_D$, let

$$U^\sigma(D) = \left\langle \prod_{\rho_k \subset \gamma \in \Sigma} D_k : \tilde{\pi}(\mathrm{int}\gamma) \subset \mathrm{int}\sigma \right\rangle$$

be the ideal in $\mathbb{C}[D_1, \ldots, D_n]$. Define the bigraded ring

$$A_1^\sigma(D)_{*,*} = U^\sigma(D)/\{u \in U^\sigma(D) : uvX^{i(\sigma)} \in (P(\Sigma) + SR(\Sigma)) \text{ for all } v \in U^\sigma(D)\},$$

where D_k has the degree (1,1).

Here is our main result.

Theorem 2.11. *Let $X \subset \mathbf{P}_\Sigma$ be an i-semiample nondegenerate hypersurface defined by $f \in S_\beta$. Then there is a ring isomorphism*

$$\bigoplus_{p,q} H^{p,q}(X) \cong \bigoplus_{p,q} A_1(X)_{p,q} \oplus \left(\bigoplus_{\sigma \in \Sigma_X} A_1^\sigma(X)_{s,s} \otimes R_1^\sigma(f)_{(q-s+1)\beta - \beta_0 + \beta_1^\sigma} \right),$$

where $s = (p + q - i - \dim \sigma + 1)/2$. The product structure on the right side is given by:
(a) $a \cdot b = ab$ for $a, b \in A_1(X)$,
(b) $a \cdot (u \otimes g) = (au) \otimes g$ for $a \in A_1(X)$, $u \in A_1^\sigma(X)$, $g \in R_1^\sigma(f)_{*\beta - \beta_0 + \beta_1^\sigma}$,

(c) $(u \otimes g) \cdot (v \otimes h) = c_r^\sigma \operatorname{Res}^\sigma(gh) X^{i(\sigma)-1} uv$ for $\dim \sigma < i - 1$, $u, v \in A_1^\sigma(X)$, $g \in R_1^\sigma(f)_{(r+1)\beta - \beta_0 + \beta_1^\sigma}$, $h \in R_1^\sigma(f)_{t\beta - \beta_0 + \beta_1^\sigma}$ ($\operatorname{Res}^\sigma(gh)$ vanishes unless $r + t = i(\sigma)$, $i(\sigma) = i - \dim \sigma$),

(d) $(u \otimes g) \cdot (v \otimes h) = -\operatorname{Res}^\sigma(gh) uv + (uv) \otimes p_\sigma(gh)$ for $\dim \sigma = i - 1$, u, v, g, h, as in part (b),

(e) $(u \otimes g) \cdot (v \otimes h) = 0$ if $u \otimes g$ and $v \otimes h$ belong to the factors in the decomposition, corresponding to distinct cones in Σ_X.

Proof. The first factor $A_1(X)_{p,q}$ in the decomposition of $H^{p,q}(X)$ is the toric part of the cohomology. We will prove that the rest is isomorphic to the residue part.

Let $u \in U^\sigma(X)$ and $g \in R_1^\sigma(f)_{*\beta - \beta_0 + \beta_1^\sigma}$. By the definition of $U^\sigma(X)$, we can assume that $u = b \cdot \prod_{\rho_k \subset \gamma} i^* D_k$ for some $b \in U^\sigma(X)$ and $\gamma \in \Sigma$ such that $\tilde{\pi}(\mathrm{int}\gamma_k) \subset \mathrm{int}\sigma$. The map assigning $b\varphi_{\gamma!}\operatorname{Res}_\gamma(g)$ to $u \otimes g$ is well defined, because

$$\frac{\varphi_{\gamma!}\operatorname{Res}_\gamma(A)}{\operatorname{mult}(\gamma)} = \prod_{\rho_k \subset \gamma' \setminus \gamma} i^* D_k \varphi_{\gamma'!}\operatorname{Res}_{\gamma'}(A)$$

for any $\gamma' \subset \gamma$ such that $\tilde{\pi}(\mathrm{int}\gamma') \subset \mathrm{int}\sigma$ and $\tilde{\pi}(\mathrm{int}\gamma) \subset \mathrm{int}\sigma$. To show this use the Gysin projection formula as in the proof of Proposition 2.9. We omit these details. Hence, by Proposition 2.8, we have a surjective map

$$\bigoplus_{\sigma \in \Sigma_X} U^\sigma(X)_{s,s} \otimes R_1^\sigma(f)_{(q-s+1)\beta - \beta_0 + \beta_1^\sigma} \xrightarrow{\psi} H_{\mathrm{res}}^{p,q}(X), \qquad (14)$$

where $s = (p + q - i - \dim \sigma + 1)/2$. To find the kernel of this map we use the Poincaré duality. This duality and Proposition 2.9(iii) show that the kernel cannot intersect the sum of two or more factors on the left side of (14). Therefore, we can assume that a given element from the kernel belongs to one of the factors $U^\sigma(X)_{s,s} \otimes R_1^\sigma(f)_{(q-s+1)\beta - \beta_0 + \beta_1^\sigma}$ for some $\sigma \in \Sigma_X$. Let $\sum_{k=1}^l u_k \otimes g_k$ be this element, where $g_1, \dots, g_l$ is a basis of $R_1^\sigma(f)_{(q-s+1)\beta - \beta_0 + \beta_1^\sigma}$, and $u_k \in U^\sigma(X)_{s,s}$. The composition

$$R_1^\sigma(f)_{(q-s+1)\beta - \beta_0 + \beta_1^\sigma} \otimes R_1^\sigma(f)_{(i(\sigma)-q+s)\beta - \beta_0 + \beta_1^\sigma} \to R_1^\sigma(f)_{(i(\sigma)+1)\beta - 2\beta_0 + 2\beta_1^\sigma} \xrightarrow{\operatorname{Res}^\sigma} \mathbb{C},$$

where the first arrow is a multiplication, is a nondegenerate pairing because of Proposition 2.2(ii), [M1, Theorem 4.4] and the definition of $\operatorname{Res}^\sigma$. Hence, there is a basis $g_1', \dots, g_l'$ of $R_1^\sigma(f)_{(i(\sigma)-q+s)\beta - \beta_0 + \beta_1^\sigma}$, orthogonal to $g_1, \dots, g_l$ with respect to this pairing. The fact that $\psi(\sum_{k=1}^l u_k \otimes g_k) = 0$ is equivalent to $\int_X \psi(\sum_{k=1}^l u_k \otimes g_k) \cup t = 0$ for all $t \in H^*(X)$. This is true for $t \in H_{\mathrm{toric}}^*(X)$, and, by Proposition 2.9(iii), we can consider only $t = \psi(v \otimes g_k')$ with $v \in U^\sigma(X)_{d-i(\sigma)-s,d-i(\sigma)-s}$ and $k = 1, \dots, l$. Applying the parts (i) and (ii) of Proposition 2.9, we get

$$\int_X \psi(\sum_{k=1}^l u_k \otimes g_k) \cup \psi(v \otimes g_k') = \int_{\mathbf{P}_\Sigma} c_{q-s}^\sigma \operatorname{Res}^\sigma(g_k g_k') X^{i(\sigma)} u_k v.$$

Hence, $\psi(\sum_{k=1}^l u_k \otimes g_k) = 0$ if and only if $X^{i(\sigma)} u_k v \in (P(\Sigma) + SR(\Sigma))$ for all k and $v \in U^\sigma(X)_{d-i(\sigma)-s,d-i(\sigma)-s}$. In the last condition we can take all $v \in U^\sigma(X)$, because the cohomology $H^*(\mathbf{P}_\Sigma) \cong \mathbb{C}[D_1, \dots, D_n]/(P(\Sigma) + SR(\Sigma))$ of the complete simplicial toric variety $\mathbf{P}_\Sigma$ satisfies the Poincaré duality. Thus, the decomposition of $H^{*,*}(X)$ is proved, and the product structure on it follows from Proposition 2.9. $\square$

3. The Picard group of semiample hypersurfaces.

In this section we will explicitly describe the Picard group of semiample nondegenerate hypersurfaces in a complete simplicial toric variety. Our approach provides a better description than the one in [R] with complicated restrictions.

First, we have the following property for semiample nondegenerate hypersurfaces.

Lemma 3.1. *Let X be an i-semiample nondegenerate hypersurface in a complete simplicial toric variety $\mathbf{P}_\Sigma$. Then*

$$H^k(X, \mathcal{O}_X) = 0 \quad \text{for } k \neq 0, i-1.$$

Proof. Since

$$H^k(X, \mathcal{O}_X) \cong H^{0,k}(X) \cong H^{0,k}_{\text{toric}}(X) \oplus H^{0,k}_{\text{res}}(X)$$

and $H^{0,k}(\mathbf{P}_\Sigma)$ vanishes for $k \neq 0$, the statement is implied by Proposition 2.7. $\square$

By this lemma, for an i-semiample nondegenerate hypersurface X with $i > 3$, we have

$$H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0.$$

Combining this with the standard exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

is enough to conclude

$$\text{Pic}(X)_\mathbb{C} \cong H^2(X, \mathbb{C}) \cong H^{1,1}(X).$$

Theorem 3.2. *Let $X \subset \mathbf{P}_\Sigma$ be an i-semiample nondegenerate hypersurface defined by $f \in S_\beta$, and let $i > 3$. Then*

$$\text{Pic}(X)_\mathbb{C} \cong H^2(X, \mathbb{C}) \cong H^{1,1}(X)$$

$$\cong \left(\bigoplus_{k=1}^n \mathbb{C}D_k \right) / C \bigoplus \left(\bigoplus_{\sigma \in \Sigma_X(i-1)} \bigoplus_{\tilde{\pi}(\text{int}\rho_k) \subset \text{int}\sigma} \mathbb{C}D_k \otimes R_1^\sigma(f)_{\beta - \beta_0 + \beta_1^\sigma} \right),$$

where $C := \text{span}_\mathbb{C}\left\{ \sum_{k=1}^n \langle m, e_k \rangle D_k, D_l : m \in M, \tilde{\pi}(\text{int}\rho_l) \subset \text{int}\sigma, \sigma \in \Sigma_X(i) \right\}$.

Proof. By Theorem 2.11,

$$H^{1,1}(X) \cong A_1(X)_{1,1} \oplus \left(\bigoplus_{\sigma \in \Sigma_X(i-1)} A_1^\sigma(X)_{1,1} \otimes R_1^\sigma(f)_{\beta - \beta_0 + \beta_1^\sigma} \right).$$

One may try to use only the definitions of $A_1(X)$ and $A_1^\sigma(X)$ to deduce the result. However, we will use some additional information to alleviate this task. By definition, $A_1(X)_{1,1}$ is a quotient of $H^{1,1}(\mathbf{P}_\Sigma) \cong \text{Pic}(\mathbf{P}_\Sigma)_\mathbb{C}$, the description of which we know, for instance, from [F]. Also, Proposition 1.6 or Lemma 1.5 imply that $X \cdot D_l = 0$ in $H^*(\mathbf{P}_\Sigma)$ for $\tilde{\pi}(\text{int}\rho_l) \subset \text{int}\sigma$, $\sigma \in \Sigma_X(i)$. This shows that $H^{1,1}(X)$ is at least a quotient space of the given answer. To prove this answer it now suffices to verify that $h^{1,1}(X)$ coincides with

$$n - d - \sum_{\sigma \in \Sigma_X(i)} a_1(\sigma) + \sum_{\sigma \in \Sigma_X(i-1)} a_1(\sigma) \dim R_1^\sigma(f)_{\beta - \beta_0 + \beta_1^\sigma},$$

where $a_1(\sigma)$ is the number of the 1-dimensional cones ρ_l such that $\tilde{\pi}(\text{int}\rho_l) \subset \text{int}\sigma$. As in [M1, Section 4], the Gysin spectral sequence gives the exact sequence:

$$0 \to H^{1,1}(H^1(X \cap \mathbf{T})) \to \bigoplus_{k=1}^{n} H^{0,0}(X \cap D_k) \to H^{1,1}(X) \to H^{1,1}(H^2(X \cap \mathbf{T})) \to 0.$$

Since $i > 3$, $H^{1,1}(H^2(X \cap \mathbf{T}))$ vanishes as in the proof of Proposition 2.7(i). On the other side, Theorem 1.7 gives $h^{1,1}(H^1(X \cap \mathbf{T})) = h^{1,1}(H^1(\mathbf{T})) = d$. We also have

$$H^0(X \cap D_k) = H^0_{\text{toric}}(X \cap D_k) \oplus H^0_{\text{res}}(X \cap D_k).$$

The dimension of $H^0_{\text{toric}}(X \cap D_k)$ is clearly 1, when $X \cap D_k$ is nonempty. By Proposition 1.6, the last condition is the same as $\tilde{\pi}(\text{int}\rho_k) \subset \text{int}\sigma$ for some $\sigma \in \Sigma_X$ of dimension less than i. Finally, Proposition 2.7 gives $H^0_{\text{res}}(X \cap D_k) \cong R_1^\sigma(f)_{\beta - \beta_0 + \beta_1^\sigma}$, and a simple count verifies the dimension of $H^{1,1}(X)$. $\qquad\square$

It would actually be convenient for us to use a more geometric description of the Picard group of X. The factor $\mathbb{C}D_k \otimes R_1^\sigma(f)_{\beta - \beta_0 + \beta_1^\sigma}$ in the above theorem is isomorphic to $H^0_{\text{res}}(X \cap D_k)$, while there is an isomorphism

$$\pi^*_{\rho_k} : H^0_{\text{res}}(Y \cap V(\sigma)) \cong H^0_{\text{res}}(X \cap D_k).$$

The intersection $Y \cap V(\sigma)$ is a finite set of points $p_1, \ldots, p_{\text{vol}(\Gamma_\sigma)}$ inside the 1-dimensional toric variety $V(\sigma) \cong \mathbb{P}^1$. Hence,

$$H^0(Y \cap V(\sigma)) \cong \bigoplus_{l=1}^{\text{vol}(\Gamma_\sigma)} \mathbb{C}p_l.$$

It follows from the definition of the residue part that $H^0_{\text{res}}(Y \cap V(\sigma))$ consists of $\sum_{l=1}^{\text{vol}(\Gamma_\sigma)} a_l p_l$ such that $\sum_{l=1}^{\text{vol}(\Gamma_\sigma)} a_l = 0$. Correspondingly, we get that $X \cap D_k = \bigcup_{l=1}^{\text{vol}(\Gamma_\sigma)} X_{k,l}$ is a disjoint union of $\text{vol}(\Gamma_\sigma)$ irreducible components, and

$$H^0_{\text{res}}(X \cap D_k) \cong \left\{ \sum_{l=1}^{\text{vol}(\Gamma_\sigma)} a_l X_{k,l} : \sum_{l=1}^{\text{vol}(\Gamma_\sigma)} a_l = 0 \right\}.$$

Corollary 3.3. *Let $X \subset \mathbf{P}_\Sigma$ be an i-semiample nondegenerate hypersurface with $i > 3$. Then*

$$\text{Pic}(X)_{\mathbb{C}} \cong \left(\bigoplus_{k=1}^{n} \mathbb{C}D_k \right) / C \bigoplus \left(\bigoplus_{\substack{\sigma \in \Sigma_X(i-1) \\ \tilde{\pi}(\text{int}\rho_k) \subset \text{int}\sigma}} \left\{ \sum_{l=1}^{\text{vol}(\Gamma_\sigma)} a_l X_{k,l} : \sum_{l=1}^{\text{vol}(\Gamma_\sigma)} a_l = 0 \right\} \right),$$

where C is as in Theorem 3.2, and $X_{k,l}$ is a connected component of $X \cap D_k$.

To connect this description of the Picard group to the one in Theorem 3.2, we will show which elements in the latter correspond to $-X_{k,l} + X_{k,l+1}$, $l = 1, \ldots, \text{vol}(\Gamma_\sigma) - 1$, in the former. Why we choose such a basis will be clear when we construct a mirror map in Section 5.

Let X be determined by $f \in S_\beta$ be linearly equivalent to a torus invariant divisor $\sum_{k=1}^{n} b_k D_k$ with the associated polytope $\Delta_D = \{m \in M_{\mathbb{R}} : \langle m, e_k \rangle \geq -b_k$ for all $k\}$. Write $f = \sum_{m \in \Delta \cap M} a_m x^{D(m)}$ where $x^{D(m)} = \prod_{k=1}^{n} x_k^{b_k + \langle m, e_k \rangle}$. The restriction of this polynomial $f|_{D_k} = \sum_{m \in \Gamma_\sigma \cap M} a_m x^{D(m)}$ determines $X \cap D_k$. Since m_1^σ is a basis element for the lattice $M_X \cap \sigma^\perp$, the lattice points $m \in \Gamma_\sigma$ are

$m_0 + sm_1^\sigma$, $s = 0, \ldots, \mathrm{vol}(\Gamma_\sigma)$, where m_0 is some of the vertices of Γ_σ. Then the polynomial $f|_{D_k}$ can be factored as

$$a_{m_0+\mathrm{vol}(\Gamma_\sigma)m_1^\sigma} x^{D(m_0)} \prod_{s=1}^{\mathrm{vol}(\Gamma_\sigma)} (\prod_{k=1}^n x_k^{\langle m_1^\sigma, e_k \rangle} - \lambda_s),$$

where $\prod_{k=1}^n x_k^{\langle m_1^\sigma, e_k \rangle}$ is a coordinate on the torus $\mathbf{T}_{\rho_k}$, which is equal to λ_s precisely on the connected component $X_{k,s}$ of $X \cap D_k$. This can be seen from the description of the map π in Proposition 1.6.

Lemma 3.4. *Let $X \subset \mathbf{P}_\Sigma$ be an i-semiample nondegenerate hypersurface defined by $f \in S_\beta$. Then $-X_{k,l} + X_{k,l+1}$, $l = 1, \ldots, \mathrm{vol}(\Gamma_\sigma) - 1$, corresponds to $D_k \otimes g_l$, where*

$$g_l = \frac{(\lambda_{l+1} - \lambda_l) f \prod_{k=1}^n x_k^{\langle m_1^\sigma, e_k \rangle}}{(\prod_{k=1}^n x_k^{\langle m_1^\sigma, e_k \rangle} - \lambda_l)(\prod_{k=1}^n x_k^{\langle m_1^\sigma, e_k \rangle} - \lambda_{l+1}) \prod_{\tilde{\pi}(\rho_k) \not\subset \sigma} x_k} \in R_1^\sigma(f)_{\beta - \beta_0 + \beta_1^\sigma}.$$

Proof. It suffices to show that the class of $-X_{k,l} + X_{k,l+1}$ in $H^{0,0}(X \cap D_k)$ is represented by the cocycle

$$\left\{ \frac{(\lambda_{l+1} - \lambda_l) f \prod_{k=1}^n x_k^{\langle m_1^\sigma, e_k \rangle} \langle m_1^\sigma, e_{i_0} \rangle}{(\prod_{k=1}^n x_k^{\langle m_1^\sigma, e_k \rangle} - \lambda_l)(\prod_{k=1}^n x_k^{\langle m_1^\sigma, e_k \rangle} - \lambda_{l+1}) x_{i_0} f_{i_0}} \right\}_{i_0},$$

which also represents $\mathrm{Res}_{\rho_k}(g_l)$, by Proposition 2.3. Note that

$$x_{i_0} f_{i_0}|_{D_k} = a_{m_0+\mathrm{vol}(\Gamma_\sigma)m_1^\sigma} x^{D(m_0)} \Bigg(\Big(b_{i_0} + \langle m_0, e_{i_0} \rangle \Big) \prod_{s=1}^{\mathrm{vol}(\Gamma_\sigma)} (\prod_{k=1}^n x_k^{\langle m_1^\sigma, e_k \rangle} - \lambda_s)$$

$$+ \langle m_1^\sigma, e_{i_0} \rangle \prod_{k=1}^n x_k^{\langle m_1^\sigma, e_k \rangle} \sum_{j=1}^{\mathrm{vol}(\Gamma_\sigma)} \prod_{s \neq j} (\prod_{k=1}^n x_k^{\langle m_1^\sigma, e_k \rangle} - \lambda_s) \Bigg).$$

Since $\prod_{k=1}^n x_k^{\langle m_1^\sigma, e_k \rangle}$ has the value λ_s on $X_{k,s}$, the above cocycle is equal to -1 on $X_{k,l}$, 1 on $X_{k,l+1}$ and 0 on all other components. $\qquad\square$

In Section 5, we will describe the product structure on the ring generated by $\mathrm{Pic}(X)_\mathbb{C}$.

4. The B-model correlation functions.

In [M2], we calculated a subring of the B-model chiral ring $H^*(X, \wedge^* \mathcal{T}_X)$, which contains the space $H^1(X, \mathcal{T}_X)$, for semiample anticanonical nondegenerate hypersurfaces. Here, we notice that the ring structure for minimal hypersurfaces is related to a product of the roots of A-type Lie algebra. Let us note that a semiample anticanonical nondegenerate hypersurface $X \subset \mathbf{P}_\Sigma$ is big and Calabi-Yau. We will use the products on the chiral ring in the next section to propose a mirror map between the space $H^1(X, \mathcal{T}_X)$ and the Picard group of the mirror symmetric Calabi-Yau hypersurface X° in the Batyrev construction.

We have the birational contraction $\pi : \mathbf{P}_\Sigma \to \mathbf{P}_{\Sigma_X}$. In [M2, Section 4], we ordered the cones ρ_i inside $\sigma \in \Sigma_X(2)$, according the way they lie in σ: $\rho_{l_0}, \ldots, \rho_{l_{n(\sigma)+1}}$, so that ρ_{l_0} and $\rho_{l_{n(\sigma)+1}}$ are the edges of σ. The corresponding divisors D_{l_k} in $\mathbf{P}_\Sigma$ intersect only if they are next to each other in the order. Using this order we can state [M2, Theorem 7.1].

Theorem 4.1. [M2] *Let $X \subset \mathbf{P}_\Sigma$ be a semiample anticanonical nondegenerate hypersurface defined by $f \in S_\beta$. Then there is a natural inclusion*

$$\gamma_- \oplus (\oplus \gamma_-^{\sigma,k}) : R_1(f)_{*\beta} \oplus \left(\bigoplus_{\sigma \in \Sigma_X(2)} \left(R_1^\sigma(f)_{(*-1)\beta+\beta_1^\sigma} \right)^{n(\sigma)} \right) \hookrightarrow H^*(X, \wedge^* \mathcal{T}_X),$$

where the sum $\oplus \gamma_-^{\sigma,k}$ is over ρ_{l_k}, $k = 1, \ldots, n(\sigma)$. Also, $R_1^\sigma(f)_{(q-1)\beta+\beta_1^\sigma} = 0$ for $q = 0, d-1$.

To simplify our further calculations, in this section we make an additional assumption on $\mathbf{P}_\Sigma$:

$$\mathrm{mult}(\sigma') = 1 \text{ for all } \sigma' \in \Sigma(2) \text{ such that } \sigma' \subset \sigma \in \Sigma_X(2). \tag{15}$$

In particular, this holds if $X \subset \mathbf{P}_\Sigma$ is a minimal Calabi-Yau hypersurface in the mirror construction of [B]. There is a nice way to state the product structure on the subspace of $H^*(X, \wedge^* \mathcal{T}_X)$ in the above theorem. For this we need to use the $(n(\sigma)+1)$-dimensional vectors $\alpha_k = (0, \ldots, 0, -1, 1, 0, \ldots, 0)$ with -1 on the k-th place, which form a basis of the root system of $A_{n(\sigma)}$-type Lie algebra. Also, define the following product of an arbitrary number of vectors $v_l = (v_{l,1}, \ldots, v_{l,n(\sigma)+1}) \in \mathbb{R}^{n(\sigma)+1}$, $l = 1, \ldots, L$, by the rule:

$$v_1 \cdots v_L = \sum_{k=1}^{n(\sigma)+1} v_{1,k} \cdots v_{L,k}.$$

As a corollary of [M2, Theorem 6.3], we have:

Theorem 4.2. *Let $X \subset \mathbf{P}_\Sigma$ be a semiample anticanonical nondegenerate hypersurface defined by $f \in S_\beta$, and assume (15) holds. Then, under the identifications of Theorem 4.1, we have*
  (i) *$\gamma_A \cup \gamma_B = \gamma_{AB}$,*
  (ii) *$\gamma_A \cup \gamma_B^{\sigma,k} = \gamma_{AB}^{\sigma,k}$,*
  (iii) *$\gamma_A^{\sigma_1,k_1} \cup \gamma_B^{\sigma_2,k_2} = 0$ if $\sigma_1 \neq \sigma_2$,*
  (iv) *for $A, B \in R_1^\sigma(f)_{(*-1)\beta+\beta_1^\sigma}$, such that $AB \in R_1^\sigma(f)_{(d-3)\beta+2\beta_1^\sigma}$,*

$$\gamma_A^{\sigma,k_1} \cup \gamma_B^{\sigma,k_2} = \alpha_{k_1} \cdot \alpha_{k_2} \gamma_{\mu^{-1}(ABG^\sigma(f))} \in H^{d-1}(X, \wedge^{d-1} \mathcal{T}_X),$$

where $\mu : R_1(f)_{(d-1)\beta} \to R_0(f)_{d\beta}$ is an isomorphism defined by multiplication with $\prod_{k=1}^n x_k$, and where $G^\sigma(f) := \dfrac{x_{l_0} f_{l_0} x_{l_{n(\sigma)+1}} f_{l_{n(\sigma)+1}} \prod_{\rho_k \not\subset \sigma} x_k}{\mathrm{mult}(\sigma) \prod_{\rho_k \subset \sigma} x_k} \in S_{3\beta - 2\beta_1^\sigma}$,
  (v) *for $A, B \in R_1^\sigma(f)_{(*-1)\beta+\beta_1^\sigma}$, such that $ABC \in R_1^\sigma(f)_{(d-4)\beta+3\beta_1^\sigma}$,*

$$\gamma_A^{\sigma,k_1} \cup \gamma_B^{\sigma,k_2} \cup \gamma_C^{\sigma,k_3} = \alpha_{k_1} \cdot \alpha_{k_2} \cdot \alpha_{k_3} \gamma_{\mu^{-1}(ABCH^\sigma(f)G^\sigma(f))} \in H^{d-1}(X, \wedge^{d-1} \mathcal{T}_X),$$

where

$$H^\sigma(f) := \sqrt{-1} \sum_{m \in \sigma^\perp \cap \Delta \cap M} a_m \frac{x^{D(m)}}{\prod_{\rho_k \subset \sigma} x_k} \in S_{\beta - \beta_1^\sigma}$$

with $\sigma \in \Sigma_X(2)$, $f = \sum_{m \in \Delta \cap M} a_m x^{D(m)}$, and the polytope Δ corresponding to the anticanonical divisor $\sum_{k=1}^n D_k$.

We can now see a peculiar product structure on a part of the B-model chiral ring $\oplus_{p,q} H^p(X, \wedge^q \mathcal{T}_X)$. This ring contains modules $\oplus_q R_1^\sigma(f)_{(q-1)\beta+\beta_1^\sigma}$, corresponding to $e_k \in \mathrm{int}\sigma$, over the subring $\oplus_q R_1(f)_{q\beta}$ called the polynomial part in [M2]. The

product structure on the sum of these modules is governed by a product of the roots of $A_{n(\sigma)}$-type Lie algebra. While this appears to be surprising, one likely reason for such a mystery comes from the fact that the contraction $Y = \pi(X)$ has $A_{n(\sigma)}$-type singularities along a codimension 2 subvariety $Y \cap V(\sigma)$. The generators $e_k \in \text{int}\,\sigma$ correspond to the exceptional divisors D_k of the blow up π, whose intersection matrix forms the Dynkin diagram of $A_{n(\sigma)}$, as it was observed in [KaMoP] for the 3-dimensional Calabi-Yau hypersurfaces.

While it is difficult to compute all of the $(d-1)$-point functions of the Calabi-Yau hypersurface, the following products follow from Theorem 4.2.

Corollary 4.3. *Let $X \subset \mathbf{P}_\Sigma$ be a semiample anticanonical nondegenerate hypersurface defined by $f \in S_\beta$, and assume (15) holds. Given $A_1, \ldots, A_s \in R_1(f)_\beta$ and $B_1, \ldots, B_t \in R_1^\sigma(f)_{\beta_1^\sigma}$, $s + t = d - 1$, $t \leq 3$, then*

$$\gamma_{A_1} \cup \cdots \cup \gamma_{A_s} \cup \gamma_{B_1}^{\sigma,k_1} \cup \cdots \cup \gamma_{B_t}^{\sigma,k_t} = \alpha_{k_1} \cdots \alpha_{k_t} \gamma_{\mu^{-1}(A_1 \cdots A_s B_1 \cdots B_t H^\sigma(f)^{t-2} G^\sigma(f))}.$$

This corollary together with the products on the polynomial part $\oplus_q R_1(f)_{q\beta}$ of the B-model chiral ring, gives all information about the non-normalized 3-point functions (Yukawa couplings) for the Calabi-Yau 3-fold hypersurfaces.

5. A GENERALIZATION OF THE MONOMIAL-DIVISOR MIRROR MAP.

The monomial-divisor mirror map was proposed in [AGM]. This map was conjectured to be the derivative of the mirror map at large radius limit points (maximally unipotent boundary points) between the complex and Kähler moduli spaces of a mirror pair (X, X°) of Calabi-Yau hypersurfaces in toric varieties. Here we extend this map to the whole $\text{Pic}(X)_{\mathbb{C}}$ and $H^1(X^\circ, \mathcal{T}_{X^\circ})$. Our map is supported by a compatibility of some limiting products of the chiral rings.

While Theorem 2.11 already has a description of the products on the $\text{Pic}(X)_{\mathbb{C}}$, it is very technical to compute Res^σ and p_σ directly. That is why we found an alternative description of the Picard group of X in Corollary 3.3. Using this description we will prove the following.

Proposition 5.1. *Let $X \subset \mathbf{P}_\Sigma$ be an i-semiample nondegenerate hypersurface defined by $f \in S_\beta$. Given $a_1, \ldots, a_s \in H^{1,1}_{\text{toric}}(X)$, then*

$$a_1 \cup \cdots \cup a_s \cup (D_{i_1} \otimes g_{k_1}) \cup \cdots \cup (D_{i_t} \otimes g_{k_t}) = \alpha_{k_1} \cdots \alpha_{k_t} a_1 \cdots a_s \cdot D_{i_1} \cdots D_{i_t},$$

where $s + t = d - 1$.

Proof. By Lemma 3.4, $D_{i_j} \otimes g_{k_j}$ corresponds to $-X_{i_j,k_j} + X_{i_j,k_j+1}$, which is the same as $\varphi_{\rho_{i_j}!} \pi^*_{\rho_{i_j}}(-p_{k_j} + p_{k_j+1})$. Let us remind that

$$Y \cap V(\sigma) = \pi(X) \cap V(\sigma) = \bigcup_{k=1}^{\text{vol}(\Gamma_\sigma)} p_k.$$

The product structure on

$$H^0(Y \cap V(\sigma)) \cong \bigoplus_{k=1}^{\text{vol}(\Gamma_\sigma)} \mathbb{C} p_l$$

is an obvious one: multiplication of the corresponding components of the vectors. It is not difficult to see that this product can be written as

$$(-p_{k_1} + p_{k_1+1}) \cup (-p_{k_2} + p_{k_2+1}) = \frac{\alpha_{k_1} \cdot \alpha_{k_2}}{n(\sigma) + 1} \sum_{k=1}^{\mathrm{vol}(\Gamma_\sigma)} p_k + \sum_{k=1}^{n(\sigma)} c_k(-p_k + p_{k+1})$$

where the coefficients c_k are determined by the decomposition

$$(\alpha_{k_1,1}\alpha_{k_2,1}, \ldots, \alpha_{k_1,n(\sigma)+1}\alpha_{k_2,n(\sigma)+1}) = \frac{\alpha_{k_1} \cdot \alpha_{k_2}}{n(\sigma) + 1}(1, \ldots, 1) + \sum_{k=1}^{n(\sigma)} c_k \alpha_k.$$

Hence,

$$(D_{i_1} \otimes g_{k_1}) \cup (D_{i_2} \otimes g_{k_2}) = \varphi_{\rho_{i_1}!} \pi_{\rho_{i_1}}^* (-p_{k_1} + p_{k_1+1}) \cup \varphi_{\rho_{i_2}!} \pi_{\rho_{i_2}}^* (-p_{k_2} + p_{k_2+1})$$

$$= i^* D_{i_2} \cup \varphi_{\rho_{i_1}!} \pi_{\rho_{i_1}}^* ((-p_{k_1} + p_{k_1+1}) \cup (-p_{k_2} + p_{k_2+1}))$$

$$= \frac{\alpha_{k_1} \cdot \alpha_{k_2}}{n(\sigma) + 1} D_{i_1} D_{i_2} + \sum_{k=1}^{n(\sigma)} c_k(D_{i_1} D_{i_2} \otimes g_k),$$

where we also use the same technique as in the proof of Proposition 2.9.

Now we can prove the proposition by induction on t. It is true for $t = 1$: the left side of the formula vanishes because the residue part is orthogonal to the toric part, while the other side also vanishes by our definition of the products of the vectors. Assuming that the statement holds for $t - 1$, we have

$$a_1 \cup \cdots \cup a_s \cup (D_{i_1} \otimes g_{k_1}) \cup \cdots \cup (D_{i_t} \otimes g_{k_t})$$

$$= a_1 \cup \cdots \cup a_s \cup \left(\frac{\alpha_{k_1} \cdot \alpha_{k_2}}{n(\sigma) + 1} D_{i_1} D_{i_2} + \sum_{k=1}^{n(\sigma)} c_k(D_{i_1} D_{i_2} \otimes g_k) \right) \cup (D_{i_3} \otimes g_{k_3}) \cup \cdots \cup (D_{i_t} \otimes g_{k_t})$$

$$= \frac{\alpha_{k_1} \cdot \alpha_{k_2}}{n(\sigma) + 1} a_1 \cdots a_s D_{i_1} D_{i_2} \cdots D_{i_t} + \sum_{k=1}^{n(\sigma)} c_k \alpha_k \cdot \alpha_{k_3} \cdots \alpha_{k_t} a_1 \cdots a_s D_{i_1} D_{i_2} \cdots D_{i_t}$$

$$= \alpha_{k_1} \cdots \alpha_{k_t} a_1 \cdots a_s \cdot D_{i_1} \cdots D_{i_t}.$$

$$\square$$

We can see a similarity between the products in the above proposition and Corollary 4.3. Based on this we will propose a mirror map between $\mathrm{Pic}(X)_{\mathbb{C}}$ and $H^1(X^\circ, \mathcal{T}_{X^\circ})$ for a mirror pair (X, X°) of Calabi-Yau hypersurfaces.

Let us recall the mirror symmetry construction from [B]. We will describe it starting with a semiample nondegenerate Calabi-Yau hypersurface X in a complete simplicial toric variety $\mathbf{P}_\Sigma$. Such a hypersurface has the anticanonical degree $[\sum_{i=1}^n D_n]$. Theorem 1.2 gives a unique birational toric morphism $\pi : \mathbf{P}_\Sigma \rightarrow \mathbf{P}_{\Sigma_X}$ such that the fan Σ is a subdivision of the fan Σ_X, $\pi_*[X]$ is ample and $\pi^*\pi_*[X] = [X]$. Moreover, Σ_X is the normal fan of the polytope Δ associated with a torus invariant divisor equivalent to X. The push-forward $\pi_*[X]$ is again anticanonical, whence the toric variety $\mathbf{P}_{\Sigma_X}$ is Fano associated with the polytope $\Delta \subset M_{\mathbb{R}}$ of the anticanonical divisor $\sum_{i=1}^n D_n$ on $\mathbf{P}_\Sigma$, by Lemma 3.5.2 in [CKa]. Then, Proposition 1.6 shows that the image $Y := \pi(X)$ is an ample nondegenerate hypersurface in $\mathbf{P}_{\Sigma_X} = \mathbf{P}_\Delta$. The fact that $\mathbf{P}_\Delta$ is Fano means by Proposition 3.5.5 in [CKa] that the polytope Δ is reflexive, i.e., its dual

$$\Delta^\circ = \{n \in N_{\mathbb{R}} : \langle m, n \rangle \geq -1 \text{ for } m \in \Delta\}$$

has all its vertices at lattice points in N, and the only lattice point in the interior of Δ° is the origin 0. This implies that the generators of the 1-dimensional cones of the fan Σ are among the lattice points of Δ°. If X was a minimal Calabi-Yau (see [CKa]), then these generators coincide with the set $\Delta^\circ \cap N \setminus \{0\}$. Now, consider the polytope Δ° and the associated toric variety $\mathbf{P}_{\Delta^\circ}$. Theorem 4.1.9 in [B] says that an anticanonical nondegenerate hypersurface $Y^\circ \subset \mathbf{P}_{\Delta^\circ}$ is a Calabi-Yau variety with canonical singularities. To obtain a mirror of X, it is natural to take a certain desingularization of Y°. Batyrev introduced the following notion.

Definition 5.2. A projective birational morphism $\varphi : W' \to W$ is called a *maximal projective crepant partial desingularization* (MPCP-desingularization) of W if φ is crepant ($\varphi^*(K_W) = K_{W'}$) and W' has only $\mathbb{Q}$-factorial terminal singularities.

Take any simplicial subdivision Σ° of the normal fan of Δ° such that the generators of the 1-dimensional cones of Σ° is the set $\Delta \cap M \setminus \{0\}$. Such a subdivision corresponds to a toric birational morphism $\mathbf{P}_{\Sigma^\circ} \to \mathbf{P}_{\Delta^\circ}$ which is a MPCP-desingularization of $\mathbf{P}_{\Delta^\circ}$, by Lemma 4.1.2 in [CKa]. Theorem 2.2.24 of [B] says that the toric variety $\mathbf{P}_{\Delta^\circ}$ admits at least one MPCP-desingularization. This also induces a MPCP-desingularization $X^\circ \subset \mathbf{P}_{\Sigma^\circ}$ of the anticanonical hypersurface $Y^\circ \subset \mathbf{P}_{\Delta^\circ}$. The hypersurface X° is a minimal Calabi-Yau variety. Batyrev showed the following formulas for some Hodge numbers of the minimal Calabi-Yau hypersurfaces (MPCP-desingularizations):

$$h^{d-2,1}(X^\circ) = h^{1,1}(X) = l(\Delta^\circ) - d - 1 - \sum_{\text{codim}\theta=1} l^*(\theta) + \sum_{\text{codim}\theta=2} l^*(\theta)l^*(\theta^*),$$

where θ is a face of Δ°, θ^* is the corresponding dual face of the dual reflexive polyhedron Δ, and $l(\Gamma)$ (resp., $l^*(\Gamma)$) denotes the number of lattice (resp., interior lattice) points in Γ. For Calabi-Yau 3-folds, this is called the topological mirror symmetry test. Note also that since X° is Calabi-Yau, $h^{d-2,1}(X^\circ) = \dim H^1(X^\circ, \mathcal{T}_{X^\circ})$.

The monomial-divisor mirror map can be easily described if we assume that the polynomial $f \in S_\beta$ defining the hypersurface $X^\circ \subset \mathbf{P}_{\Sigma^\circ}$ has the form

$$\sum_{m \in (\Delta^\circ \cap M)_0} a_m x^{D(m)},$$

where $(\Delta^\circ \cap M)_0$ is the set of all lattice points in Δ° which do not lie in the interior of any facet of Δ°. Such a hypersurface has been called *simplified* in [AGM], [CKa]. In this case, the map between the toric part $H^{1,1}_{\text{toric}}(X)$ and the polynomial part $R_1(f)_\beta$ of $H^1(X^\circ, \mathcal{T}_{X^\circ})$ is defined by sending the divisor D_k, corresponding to the lattice point e_k in $(\Delta^\circ \cap M)_0$, to the monomial $x^{D(e_k)}$. One can verify that this map is well defined and induces an isomorphism. We should note that if the hypersurface is not simplified the mirror map is complicated to describe explicitly.

Our extension of the monomial-divisor mirror map is as follows. Assign to the divisors $-X_{k,l} + X_{k,l+1} = D_k \otimes g_l \in H^{1,1}(X)$, for $e_k \in \text{int}\sigma$, $\sigma \in \Sigma_X(d-1)$ and $l = 1, \ldots, \text{vol}(\Gamma_\sigma) - 1$, the elements $\gamma_A^{\sigma^\circ,l}$ with

$$A = \frac{x^{D(e_k)}}{\prod_{\rho_k \not\subset \sigma} x_k} \in R_1^{\sigma^\circ}(f)_{\beta_1^{\sigma^\circ}},$$

where $\sigma^\circ \in \Sigma_{X^\circ}(2)$ is the cone over the 1-dimensional face Γ_σ. Using the formula for the Hodge numbers and comparing the dimensions, one deduces that $\dim R_1^{\sigma^\circ}(f)_{\beta_1^{\sigma^\circ}}$ coincides with the number of $e_k \in \text{int}\sigma$. Therefore, our map is an

isomorphism. We conjecture that this map is the derivative of the mirror map at the large radius limit points of the Kähler and complex moduli spaces of the pair of Calabi-Yau hypersurfaces. One consequence of mirror symmetry is that the derivative of the mirror map between the moduli spaces gives a ring isomorphism between the quantum cohomology of X and the B-model chiral ring of X° for a mirror pair of smooth Calabi-Yau hypersurfaces. In particular, the compatibility of the products should also be valid at the limit points. The limit of the product of the quantum cohomology ring is the usual cup product on the cohomology. On the other hand, we have the products of the B-model chiral ring in Corollary 4.3. While it is very technical to compute explicitly the limits of non-vanishing products, the limits of the vanishing products are zero. The products in Corollary 4.3 vanish when there are 2 non-consecutive numbers among $k_1, \ldots, k_t$ or if $k_1, \ldots, k_t$ are the same number and t is odd. The same holds for the products of the cohomology ring in Proposition 5.1. This verifies the compatibility of our generalization of the monomial-divisor mirror map with some of the limiting products.

Since the vanishing products hold on the B-model chiral ring far from the limiting points in the complex moduli space, we believe that a similar statement should be true for the quantum products of the mirror Calabi-Yau hypersurface. Thus, we conjecture that

$$a_1 * \cdots * a_s * (D_{i_1} \otimes g_{k_1}) * \cdots * (D_{i_t} \otimes g_{k_t})$$
$$= a_1 * \cdots * a_s * (-X_{i_1,k_1} + X_{i_1,k_1+1}) * \cdots * (-X_{i_t,k_t} + X_{i_t,k_t+1}) = 0,$$

where $*$ denotes the small quantum product on $H^*(X)$ (see [CKa]), if $|k_a - k_b| > 1$ for some a, b or if $k_1 = \cdots = k_t$ and t is odd.

References

[AGM]   P. Aspinwall, B. Greene, D. Morrison, *The monomial-divisor mirror map*, Internat. Math. Res. Notices, no. 12 (1993), 319–337.

[B]   V. V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Algebraic Geometry **6** (1994), 493–535.

[BC]   V. V. Batyrev and D. A. Cox, *On the Hodge structure of projective hypersurfaces in toric varieties*, Duke Math. J. **75** (1994), 293–338.

[CaG]   J. Carlson and P. Griffiths, "Infinitesimal variations of Hodge structure and the global Torelli problem" in *Journées de Géométrie Algébrique d'Angers, juillet 1979*, Sijthoff and Nordhoff, Alphen aan den Rijn, Germantown, Md., 1980, pp. 51–76.

[C]   D. A. Cox, *Toric residues*, Ark. Mat., **34** (1996), 73–96.

[CKa]   D. Cox and S. Katz, *Algebraic Geometry and Mirror Symmetry*, Math. Surveys Monogr. **68**, Amer. Math. Soc., Providence, 1999.

[D]   V. Danilov, *The geometry of toric varieties*, Russian Math. Surveys **33** (1978), 97–154.

[DK]   V. Danilov and A. Khovanskii, *Newton polyhedra and an algorithm for computing Hodge-Deligne numbers*, Math. USSR-Izv. **29** (1987), 279–298.

[F]   W. Fulton, *Introduction to toric varieties*, Princeton Univ. Press, Princeton, NJ, 1993.

[G]   P. Griffiths, *On the periods of certain rational integrals I, II*, Ann. of Math. (2) **90** (1969) 460–495, 498–541.

[KaMoP]   S. Katz, D. Morrison, and M. R. Plesser, *Enhanced Gauge Symmetry in Type II String Theory*, Nuclear Phys. **B 477** (1996), no. 1, 105–140.

[M1]   A. R. Mavlyutov, *Semiample hypersurfaces in toric varieties*, Duke Math. J. **101** (2000), 85–116.

[M2]   _____, "On the chiral ring of Calabi-Yau hypersurfaces in toric varieties", preprint /math.AG/0010318

[M3]   _____, *Toric Geometry and Mirror Symmetry*, Ph.D. Dissertation, University of Massachusetts at Amherst, 2000.

[R]   S-S. Roan, *Picard groups of hypersurfaces in toric varieties*, Publ. Res. Inst. Math. Sci. **32** (1996), 797–834.

[V]   C. Vafa, *String vacua and orbifoldized LG models*, Modern Phys. Lett., **A4** (1989), no. 12, 1169–1185.

MAX-PLANCK-INSTITUT FÜR MATHEMATIK, BONN, D-53111, GERMANY.
E-mail address: `anvar@mpim-bonn.mpg.de`